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Abstract Harmonic Analysis

Volume II

Structure and Analysis for Compact Groups
Analysis on Locally Compact Abelian Groups

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Preface

This book is a continuation of Volume I of the same title [Grundlehren der mathematischen Wissenschaften, Band 115]. We constantly cite definitions and results from Volume I.[1] The textbook *Real and abstract analysis* by E. HEWITT and K. R. STROMBERG [Berlin · Göttingen · Heidelberg: Springer-Verlag 1965], which appeared between the publication of the two volumes of this work, contains many standard facts from analysis. We use this book as a convenient reference for such facts, and denote it in the text by RAAA. Most readers will have only occasional need actually to read in RAAA.

Our goal in this volume is to present the most important parts of harmonic analysis on compact groups and on locally compact Abelian groups. We deal with general locally compact groups only where they are the natural setting for what we are considering, or where one or another group provides a useful counterexample. Readers who are interested only in compact groups may read as follows: § 27, Appendix D, §§ 28—30 [omitting subheads (30.6) — (30.60) if desired], (31.22) — (31.25), §§ 32, 34—38, 44. Readers who are interested only in locally compact Abelian groups may read as follows: §§ 31—33, 39—42, selected Miscellaneous Theorems and Examples in §§ 34—38. For all readers, § 43 is interesting but optional.

Obviously we have not been able to cover all of harmonic analysis. The field, already immense, is growing rapidly at the present day. We were limited by space, by time, by our own abilities. We have presented the parts of the subject that every harmonic analyst must know: representations of compact groups; the WEYL-PETER theorem; PLANCHEREL'S theorem: WIENER'S Tauberian theorem. Beyond this, we have been guided largely by personal inclination. As the writing progressed, one question led naturally to another.

We have omitted special topics that are not needed for our main goals and that are treated in other monographs: RUDIN [10]; R. E. EDWARDS [7], [10]; KATZNELSON [3]; KAHANE and SALEM [3]; MAURIN [1]. We regret not having presented any Lie theory beyond the rudimentary facts set down in § 29. Plainly a detailed description of the continuous unitary irreducible representations of the classical compact

[1] An exception is the Bibliography: every work cited in Vol. II is listed at the end of Vol. II.

groups, going beyond *e.g.* BOERNER [1], and of the decompositions of their tensor products, would be of immense value for noncommutative harmonic analysis. The time seems not yet ripe for such an enterprise. A larger omission is our failure to study the algebra $\boldsymbol{M}(G)$ [G a compact or locally compact Abelian group]. This algebra presents many riddles, but enough is known for a full-scale treatment to be appropriate.

We could not have written this book without help. We are deeply grateful for the generous assistance offered by our friends. Valuable advice has been received from ROBERT B. BURCKEL, CLIFFORD V. COMISKY, RAOUF DOSS, ROBERT E. EDWARDS, LEE W. ERLEBACH, J. M. G. FELL, FRANZ VON KRBEK, A. JEANNE LADUKE, HORST LEPTIN, GEORGE W. MACKEY, JOHN R. MCMULLEN, WILLARD A. PARKER, RICHARD S. PIERCE, ROGER W. RICHARDSON, KARL R. STROMBERG, THOMAS A. SWANSON, EUGENE P. WIGNER, and JOHN H. WILLIAMSON. Significant contributions to the final form of the monograph were made by RICHARD ILTIS, BARRY E. JOHNSON, and DANIEL RIDER. Our special thanks are due to HERBERT S. ZUCKERMAN, who has helped us far more than anyone else.

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Seattle, Washington — EDWIN HEWITT
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August 1969

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Chapter Seven

Representations and duality of compact groups

In this chapter, we attempt to do for compact [non-Abelian] groups what we did in Chapter Six for locally compact Abelian groups. The task is a formidable one, and as the reader will see we do not give nearly so much detail about compact groups as we did for locally compact Abelian groups. A basic tool in our study is the WEYL-PETER theorem (27.40), which is the central object of § 27. In § 28, we apply representations of compact groups in several directions and also explore some interesting byways. In § 29, we study the unitary groups and present all the unitary representations of $\mathfrak{U}(2)$ and $\mathfrak{O}(3)$. Duality for compact groups is taken up in § 30.

§ 27. Unitary representations of compact groups

Like characters of a locally compact Abelian group, continuous unitary irreducible representations of a compact group[1] are the essential tool both for analyzing the structure of the group and for studying spaces and algebras of functions and measures defined on the group. In this section we prove a number of important facts about these representations. As usual, we begin with some definitions.

(27.1) Definition. Let S be a semigroup [not necessarily topological]. For a reflexive complex Banach space E, $E^{\sim}$ denotes the linear space of all bounded conjugate-linear functionals on E as defined in (22.1). Let E and E' be reflexive complex Banach spaces, and let V and V' be representations of S by bounded operators on $E^{\sim}$ and $E'^{\sim}$ respectively. A bounded linear transformation T carrying $E^{\sim}$ into $E'^{\sim}$ is called an *intertwining transformation* if

(i) $V'_x T = T V_x$

for all $x \in S$. The set of all intertwining transformations for V and V' is denoted by the symbol $\mathfrak{I}(V, V')$. It is easy to see that $\mathfrak{I}(V, V')$ is a

[1] Throughout this volume, we use the term "compact group" to mean exactly what the term says: a compact topological group that may or may not be Abelian. However, we will always have our eye on the non-Abelian case. From this point of view, compact Abelian groups are especially simple examples of compact groups.

closed linear subspace of the Banach space $\mathfrak{B}(E^{\sim}, E'^{\sim})$ consisting of all bounded linear transformations of $E^{\sim}$ into $E'^{\sim}$.[1]

If $\mathfrak{I}(V, V')$ contains a linear isometry (B.42) of $E^{\sim}$ onto $E'^{\sim}$, then V and V' are called *equivalent representations of* S and we write $V \sim V'$ [this definition is consistent with that of (21.8)].

(27.2) Note. If V and V' are equivalent representations of G by unitary operators on Hilbert spaces H and H', and if T is an isometry in $\mathfrak{I}(V, V')$, then T also preserves inner products: $\langle T\xi, T\eta\rangle = \langle \xi, \eta\rangle$ for all $\xi, \eta \in H$. This is pointed out in (B.43).

(27.3) Definition. Let G be a locally compact group. Let $\mathfrak{U}(G)$ be the set of all continuous irreducible unitary representations U of G.[2] Equivalence as defined in (27.1) is an equivalence relation in $\mathfrak{U}(G)$, and it defines in the usual way the set of all equivalence classes of representations U. We denote this set by the symbol Σ and refer to it as the *dual object of* G. Thus each $\sigma \in \Sigma$ is a set of representations of G and consists of all representations that are equivalent to some fixed representation. We frequently write $U^{(\sigma)}$ for a representation in the set σ.

(27.4) Remarks. (a) If G is locally compact and *Abelian*, then Σ is to be identified with the character group X of G, just as in (22.16.b). Two continuous characters of G are equivalent as representations if and only if they are equal. Thus each $\sigma \in \Sigma$ contains exactly one element. The algebras $\boldsymbol{M}(G)$ and $\mathfrak{L}_1(G)$, and the function spaces $\mathfrak{L}_p(G)$ $(1 < p < \infty)$, are studied largely in terms of the behavior of Fourier-Stieltjes and Fourier transforms [see (23.9)], *i.e.*, in terms of their "decompositions" with respect to irreducible representations of G.

(b) The present chapter is concerned with compact, not necessarily Abelian, groups, for which the situation is in some respects more complicated but in most respects simpler than for noncompact locally compact Abelian groups. Complications arise because not all representations in $\mathfrak{U}(G)$ are 1-dimensional if G is non-Abelian. However all representations

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[1] For $T_1, T_2 \in \mathfrak{B}(E^{\sim}, E'^{\sim})$, and $\omega \in E^{\sim}$, we define $(T_1 + T_2)(\omega) = T_1(\omega) + T_2(\omega)$; αT is defined analogously for $\alpha \in K$. The norm $\|T\|$ is defined in (B.8). With these definitions, $\mathfrak{B}(E^{\sim}, E'^{\sim})$ is evidently a complex Banach space.

[2] There is a possible cardinal number paradox involved here, which we elude as follows. For a fixed group G, there is an upper bound $\mathfrak{m}$ for the cardinal numbers of the Hilbert spaces on which G can act irreducibly. There is obviously a *set* $\mathscr{H}$ of Hilbert spaces containing a Hilbert space of each cardinal $\leq \mathfrak{m}$, and closed under the formation of subspaces, finite direct sums (B.61) and tensor products (D.14). Limiting ourselves to representations of G by operators on spaces H in $\mathscr{H}$, we plainly obtain a *set* $\mathfrak{U}(G)$. The exact details are of little interest for the purposes of the present book.

in $\mathfrak{U}(G)$ are *finite-dimensional* if G is compact (22.13)[1], and so elementary algebra can be applied to their study. The end result is a nearly complete structure theory for subalgebras of $\boldsymbol{M}_a(G)$ for a compact group G. We will point out at appropriate places the exact rôle played by compactness in this theory.

(c) We use the symbol Σ in the non-Abelian case instead of X to emphasize that we are dealing with an object that is not a group. For compact non-Abelian groups G, Σ admits two operations of an algebraic character, which are very useful in studying G. However, Σ does not form a group under these operations; see (27.51). The first of these operations is defined in (27.27), the second in (27.35).

(27.5) Notation and remarks. (a) For a compact group G and a fixed $\sigma \in \Sigma$, it is obvious that all $U^{(\sigma)} \in \sigma$ operate on Hilbert spaces of the same finite dimension. We reserve the symbol d_σ for the dimension of these representation spaces. Thus d_σ is a positive integer.

(b) Like every semigroup, a compact group G has a trivial 1-dimensional representation, namely the character identically equal to 1. We call this the *identity representation of* G, and we denote it by the symbol 1. Like all 1-dimensional representations of a semigroup, 1 is the only representation of G in its equivalence class.

(c) Let σ be any element of Σ, let $U^{(\sigma)}$ be in σ, and let $\{\zeta_1, \zeta_2, \ldots, \zeta_{d_\sigma}\}$ be a fixed but arbitrary orthonormal basis in the representation space H_σ of $U^{(\sigma)}$. For $j, k \in \{1, 2, \ldots, d_\sigma\}$, let $u_{jk}^{(\sigma)}$ be the function on G defined by

$$u_{jk}^{(\sigma)}(x) = \langle U_x^{(\sigma)} \zeta_k, \zeta_j \rangle$$

for $x \in G$. The functions $u_{jk}^{(\sigma)}$ are called the *coordinate functions for* $U^{(\sigma)} \in \sigma$ *and the basis* $\{\zeta_1, \zeta_2, \ldots, \zeta_{d_\sigma}\}$. Note that $\left(u_{jk}^{(\sigma)}(x)\right)_{j,k=1}^{d_\sigma}$ is the matrix of the operator $U_x^{(\sigma)}$ in this basis.

(d) Let V be an arbitrary finite-dimensional representation of an arbitrary semigroup S, with representation space H. Let $\{\xi_1, \xi_2, \ldots, \xi_d\}$ be any basis in H. Write

(i) $V_x \xi_k = \sum_{r=1}^{d} v_{rk}(x)\, \xi_r$;

thus $\left(v_{jk}(x)\right)_{j,k=1}^{d}$ is the matrix corresponding to the operator V_x in the basis $\{\xi_1, \xi_2, \ldots, \xi_d\}$. Since $V_{xy} = V_x V_y$ for all $x, y \in S$, we have

(ii) $$\sum_{r=1}^{d} v_{rk}(xy)\, \xi_r = V_x(V_y \xi_k) = \sum_{r=1}^{d} v_{rk}(y)\, V_x \xi_r$$
$$= \sum_{r=1}^{d} \sum_{s=1}^{d} v_{rk}(y)\, v_{sr}(x)\, \xi_s = \sum_{s=1}^{d} \left(\sum_{r=1}^{d} v_{sr}(x)\, v_{rk}(y) \right) \xi_s .$$

[1] The requirement that U in $\mathfrak{U}(G)$ be unitary amounts merely to imposing a certain inner product on the representation space of U. In fact, let V be any continuous representation of G by operators on the linear space K^d. Then (22.23.a) implies that V is unitary and continuous under an appropriate inner product on K^d.

From (ii), we infer the evident fact that

(iii) $v_{jk}(xy) = \sum_{r=1}^{d} v_{jr}(x) v_{rk}(y)$.

That is, the mapping $x \to (v_{jk}(x))_{j,k=1}^{d}$ is a representation of the semigroup S by matrices.

(e) Retaining the notation of (d), suppose that S is a group and that the operators V_x are unitary. Then we have

$$\begin{aligned}\langle V_{x^{-1}} \xi_k, \xi_j \rangle &= \langle V_x V_{x^{-1}} \xi_k, V_x \xi_j \rangle \\ &= \langle \xi_k, V_x \xi_j \rangle \\ &= \overline{\langle V_x \xi_j, \xi_k \rangle},\end{aligned}$$

that is,

(iv) $v_{jk}(x^{-1}) = \overline{v_{kj}(x)}$.

In view of (iii), (iv) simply asserts that $(v_{jk}(x))_{j,k=1}^{d}$ is a unitary matrix. Both (iii) and (iv) will be useful in the sequel, in dealing with coordinate functions $u_{jk}^{(\sigma)}$.

(f) Let G be a compact group and let V be any continuous unitary representation of G, with representation space H. For every pair of vectors $\xi, \eta \in H$, the function $x \to \langle V_x \xi, \eta \rangle$ is continuous on G (22.2). Hence this function is in $\mathfrak{L}_1(G)$ [recall that $\lambda(G) = 1$]. Whenever it is convenient to do so, we will regard $\mathfrak{L}_1(G)$ as a subset of $\boldsymbol{M}(G)$, as in (19.18). Thus functions $x \to \langle V_x \xi, \eta \rangle$ are associated with measures in $\boldsymbol{M}(G)$, may be convolved with each other and with arbitrary measures in $\boldsymbol{M}(G)$, and so on.

Throughout (27.6)—(27.42) and (27.44)—(27.46), G will denote a fixed but arbitrary compact group, and Σ will denote the dual object of G.

(27.6) Discussion. Let $U^{(1)}$ and $U^{(2)}$ be equivalent representations in $\mathfrak{U}(G)$, with representation spaces H_1 and H_2. Let A be a linear isometry of H_1 onto H_2 that is intertwining for $U^{(1)}$ and $U^{(2)}$. For arbitrary $\xi_1, \eta_1 \in H_1$ and $x \in G$, we have

$$\langle U_x^{(1)} \xi_1, \eta_1 \rangle = \langle A U_x^{(1)} \xi_1, A\eta_1 \rangle = \langle U_x^{(2)} A\xi_1, A\eta_1 \rangle.$$

This shows that every function on G having the form

$$x \to \langle U_x^{(1)} \xi_1, \eta_1 \rangle, \tag{1}$$

where $\xi_1, \eta_1 \in H_1$, also has the form

$$x \to \langle U_x^{(2)} \xi_2, \eta_2 \rangle, \tag{2}$$

where $\xi_2, \eta_2 \in H_2$. Similarly, every function of the form (2) also has the form (1). Thus the family of functions having the form (1) depends only upon the equivalence class to which $U^{(1)}$ belongs. This observation shows that the next definition is a proper one.

(27.7) Definition. For σ in Σ, we define $\mathfrak{T}_\sigma(G)$ as follows. For $U^{(\sigma)}$ in σ, with representation space H_σ, $\mathfrak{T}_\sigma(G)$ is the set of all finite complex linear combinations of functions of the form

$$x \to \langle U_x^{(\sigma)} \xi, \eta \rangle$$

as ξ, η vary over H_σ. For a subset P of Σ, we write $\mathfrak{T}_{\mathsf{P}}(G)$ for the smallest linear space of functions containing $\bigcup_{\sigma \in \mathsf{P}} \mathfrak{T}_\sigma(G)$. We write $\mathfrak{T}_\Sigma(G)$ simply as $\mathfrak{T}(G)$. Functions in $\mathfrak{T}(G)$ are called *trigonometric polynomials* [*on* G].

Note that $\mathfrak{T}(G) \subset \mathfrak{C}(G)$, since the representations $U^{(\sigma)}$ are continuous.

The term "trigonometric polynomial" is justified by the special case $G = T$. Here the functions $x \to \langle U_x \xi, \eta \rangle$ all have the form $\exp(it) \to \alpha \exp(int)$ (23.27.a), and $\mathfrak{T}(T)$ consists of all functions $\sum_{k=-n}^{n} \alpha_k \exp(ikt)$, *i.e.*, of all trigonometric polynomials in the ordinary sense. Note also that if G is Abelian, then $\mathfrak{T}(G)$ is the set of all linear combinations of continuous characters of G. Functions in $\mathfrak{T}(G)$ are also called *representative functions* by some writers.

(27.8) Remarks. (a) For $U^{(\sigma)} \in \sigma \in \Sigma$, let $\{\zeta_1, \zeta_2, \ldots, \zeta_{d_\sigma}\}$ be an orthonormal basis in the representation space H_σ of $U^{(\sigma)}$. It is easy to see that $\mathfrak{T}_\sigma(G)$ is the smallest linear subspace of $\mathfrak{C}(G)$ that contains the coordinate functions $u_{jk}^{(\sigma)}$ for $U^{(\sigma)}$ and this basis ($j, k \in \{1, 2, \ldots, d_\sigma\}$). Likewise, for $\mathsf{P} \subset \Sigma$, $\mathfrak{T}_{\mathsf{P}}(G)$ is the smallest linear subspace of $\mathfrak{C}(G)$ that contains all the coordinate functions $u_{jk}^{(\sigma)}$ as σ varies over P.

(b) The function spaces $\mathfrak{T}_\sigma(G)$ and $\mathfrak{T}_{\mathsf{P}}(G)$, where $\sigma \in \Sigma$ and $\mathsf{P} \subset \Sigma$, are left and right translation invariant subspaces of $\mathfrak{C}(G)$. [It suffices to observe that if f has the form $f(x) = \langle U_x \xi, \eta \rangle$, then ${}_af$ and f_a also have this form. Indeed, we have

$$f_a(x) = \langle U_{xa} \xi, \eta \rangle = \langle U_x (U_a \xi), \eta \rangle$$

and

$${}_af(x) = \langle U_{ax} \xi, \eta \rangle = \langle U_x \xi, U_{a^{-1}} \eta \rangle.]$$

(c) As pointed out in (23.19), continuous characters of G form an orthonormal set in $\mathfrak{L}_2(G)$. If G is Abelian, $\mathfrak{T}(G)$ is uniformly dense in $\mathfrak{C}(G)$, as shown in (23.20), and it follows from (12.10) that X is a *complete* orthonormal set in $\mathfrak{L}_2(G)$. Analogous facts hold in the non-Abelian case. The WEYL-PETER theorem [(27.40) *infra*] states that the functions $d_\sigma^{\frac{1}{2}} u_{jk}^{(\sigma)}$, as σ varies over all of Σ, form a complete orthonormal set in $\mathfrak{L}_2(G)$. As in the Abelian case, the orthogonality relations can be proved without recourse to the fact that $\mathfrak{T}(G)$ is uniformly dense in $\mathfrak{C}(G)$. We will establish these relations before setting up the machinery needed to prove [in (27.39)] that $\mathfrak{T}(G)$ is dense in $\mathfrak{C}(G)$. We begin with some simple algebraic facts.

(27.9) Theorem [SCHUR'S lemma]. *Let F be an arbitrary field. Let E_j be a vector space over F, and let $\mathfrak{M}_j$ be an irreducible set of linear operators on E_j $(j=1, 2)$.*[1] *Suppose that there is a linear transformation A carrying E_1 into E_2 such that $A\circ\mathfrak{M}_1=\mathfrak{M}_2\circ A$. Then either A is the zero transformation or A is a one-to-one transformation carrying E_1 onto E_2.*

Proof. Consider the linear subspace $A(E_1)$ of E_2. For $M_2\in\mathfrak{M}_2$, choose $M_1\in\mathfrak{M}_1$ such that $A\circ M_1=M_2\circ A$. Then we have

$$M_2\big(A(E_1)\big)=M_2\circ A(E_1)=(A\circ M_1)(E_1)=A\big(M_1(E_1)\big)\subset A(E_1). \qquad (1)$$

Thus $A(E_1)$ is a subspace of E_2 that is invariant under all operators in $\mathfrak{M}_2$, and since $\mathfrak{M}_2$ is by hypothesis irreducible, we see that $A(E_1)=\{0\}$ or $A(E_1)=E_2$. The first possibility simply means that $A=0$. If $A\neq 0$, then $A(E_1)=E_2$. To show that A is one-to-one, we consider the set N of $x_1\in E_1$ such that $A(x_1)=0$. Clearly N is a linear subspace of E_1, and $N\neq E_1$ since $A\neq 0$. For $M_1\in\mathfrak{M}_1$, we have $A\circ M_1=M_2\circ A$ for some $M_2\in\mathfrak{M}_2$ and hence

$$A\big(M_1(N)\big)=M_2\big(A(N)\big)=M_2\big(\{0\}\big)=\{0\}.$$

Thus $M_1(N)\subset N$ and N is a subspace of E_1 that is invariant under all operators in $\mathfrak{M}_1$. Since $\mathfrak{M}_1$ is irreducible, N is equal to $\{0\}$, and so A is one-to-one. □

(27.10) Corollary. *Let F be an algebraically closed field. Let E be a finite-dimensional vector space over F and let $\mathfrak{M}$ be an irreducible set of linear operators on E. Let A be a linear operator on E such that $AM=MA$ for all $M\in\mathfrak{M}$. Then $A=\alpha I$ for some $\alpha\in F$.*

Proof. There is an $\alpha\in F$ such that $(A-\alpha I)^{-1}$ does not exist. For, consider the function $x\to\det(A-xI)$ mapping F into F. This is a polynomial in x, and since F is algebraically closed it has a root α. For this α we have $\det(A-\alpha I)=0$ and so $(A-\alpha I)^{-1}$ does not exist[2]. The operator $A-\alpha I$ has the property that $(A-\alpha I)\mathfrak{M}=\mathfrak{M}(A-\alpha I)$. An application of (27.9) shows that $A-\alpha I=0$. □

We now apply (27.9) and (27.10) to unitary representations of our compact group G.

(27.11) Discussion. Let $U^{(1)}$ and $U^{(2)}$ be continuous unitary representations of a compact group G, not necessarily irreducible or even

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[1] By *irreducible* we mean here that no proper linear subspace of E_j is invariant under all operators in $\mathfrak{M}_j$. See (21.26) for the definition if E_j is a *topological* linear space.

[2] This is a thoroughly elementary fact. See for example BIRKHOFF and MACLANE [1], Chap. X, Theorem 3.

finite-dimensional. Let H_j, with inner product $\langle\,,\,\rangle_j$ and norm $\|\;\|_j$, be the representation space of $U^{(j)}$. We will always suppose that representation spaces for unitary representations are Hilbert spaces. Let B be a bounded linear transformation mapping H_1 into or onto H_2. For $\xi_j \in H_j$ $(j=1, 2)$, define

(i) $C_B(\xi_1, \xi_2) = \int\limits_G \langle U^{(2)}_{x^{-1}} B U^{(1)}_x \xi_1, \xi_2\rangle_2 \, dx.$

Here the integral is with respect to normalized Haar measure λ on G: recall that $\lambda(G)=1$ and that λ is both right and left invariant [(15.9) and (15.13)]. The integrand in (i) is continuous. To see this, write

$$\begin{aligned}
&|\langle U^{(2)}_{x^{-1}} B U^{(1)}_x \xi_1, \xi_2\rangle_2 - \langle U^{(2)}_{y^{-1}} B U^{(1)}_y \xi_1, \xi_2\rangle_2| \\
&\leqq |\langle B U^{(1)}_x \xi_1, U^{(2)}_x \xi_2\rangle_2 - \langle B U^{(1)}_x \xi_1, U^{(2)}_y \xi_2\rangle_2| \\
&\quad + |\langle B U^{(1)}_x \xi_1, U^{(2)}_y \xi_2\rangle_2 - \langle B U^{(1)}_y \xi_1, U^{(2)}_y \xi_2\rangle_2| \\
&\leqq \|B\| \cdot \|\xi_1\|_1 \cdot \|U^{(2)}_x \xi_2 - U^{(2)}_y \xi_2\|_2 + \|B\| \cdot \|U^{(1)}_x \xi_1 - U^{(1)}_y \xi_1\|_1 \cdot \|\xi_2\|_2.
\end{aligned}$$

Then apply (22.8), remembering that $U^{(1)}$ and $U^{(2)}$ are continuous representations. Hence the integral in (i) exists and is a complex number.

The functional $(\xi_1, \xi_2) \to C_B(\xi_1, \xi_2)$ is accordingly defined on $H_1 \times H_2$. For fixed ξ_1, it is bounded, additive, and conjugate-linear as a function of ξ_2. By (B.45), there is for each fixed $\xi_1 \in H_1$ an element $\eta_2 \in H_2$ such that $C_B(\xi_1, \xi_2) = \langle \eta_2, \xi_2\rangle_2$ for every $\xi_2 \in H_2$. Define the mapping A_B by $A_B \xi_1 = \eta_2$, so that we have

(ii) $C_B(\xi_1, \xi_2) = \langle A_B \xi_1, \xi_2\rangle_2.$

It is obvious that A_B is linear. Also A_B is bounded and $\|A_B\| \leqq \|B\|$, since

$$\begin{aligned}
\|A_B \xi_1\|_2^2 = \|\eta_2\|_2^2 = C_B(\xi_1, \eta_2) &\leqq \max\{\|B U^{(1)}_x \xi_1\|_2 \, \|\eta_2\|_2 : x \in G\} \\
&\leqq \|B\| \, \|\xi_1\|_1 \, \|A_B \xi_1\|_2.
\end{aligned}$$

The usefulness of the operators A_B depends upon the following fact.

(27.12) Theorem. *Notation is as in (27.11). The linear transformation A_B is an intertwining transformation for $U^{(1)}$ and $U^{(2)}$:*

(i) $A_B U^{(1)}_y = U^{(2)}_y A_B$ *for all* $y \in G$.

Proof. This follows at once from the right invariance of λ:

$$\begin{aligned}
\langle A_B U^{(1)}_y \xi_1, \xi_2\rangle_2 &= \int\limits_G \langle U^{(2)}_{x^{-1}} B U^{(1)}_x U^{(1)}_y \xi_1, \xi_2\rangle_2 \, dx \\
&= \int\limits_G \langle U^{(2)}_y U^{(2)}_{(xy)^{-1}} B U^{(1)}_{xy} \xi_1, \xi_2\rangle_2 \, dx = \int\limits_G \langle U^{(2)}_y U^{(2)}_{x^{-1}} B U^{(1)}_x \xi_1, \xi_2\rangle_2 \, dx \\
&= \int\limits_G \langle U^{(2)}_{x^{-1}} B U^{(1)}_x \xi_1, U^{(2)}_{y^{-1}} \xi_2\rangle_2 \, dx = \langle A_B \xi_1, U^{(2)}_{y^{-1}} \xi_2\rangle_2 \\
&= \langle U^{(2)}_y A_B \xi_1, \xi_2\rangle_2.
\end{aligned}$$

Since ξ_j is arbitrary in H_j, (i) is proved. □

We need one more preliminary result.

(27.13) Theorem. *For $j \in \{1, 2\}$, let $U^{(j)}$ be a unitary representation of an arbitrary group G with representation space H_j. Suppose that $U^{(1)}$ is irreducible and that there is a bounded linear space isomorphism A that carries H_1 onto H_2 and is an intertwining transformation for $U^{(1)}$ and $U^{(2)}$. Then there is a positive real number β such that βA is a linear isometry of H_1 onto H_2. Thus $U^{(1)}$ and $U^{(2)}$ are equivalent in the sense of* (27.1).

Proof. It is clear that $U^{(2)}$ is irreducible. Define the adjoint mapping $A^\sim$ of H_2 onto H_1 through the relation

$$\langle A^\sim \xi_2, \xi_1\rangle_1 = \langle \xi_2, A\xi_1\rangle_2 \tag{1}$$

for all $\xi_j \in H_j$ $(j = 1, 2)$. As in (27.11), the right side of (1) is conjugate-linear and continuous in ξ_1 for each fixed ξ_2 and so has the form $\langle \eta_1, \xi_1\rangle_1$; we take $A^\sim \xi_2 = \eta_1$. It is easy to see that $A^\sim$ is a bounded linear space isomorphism of H_2 onto H_1.

The transformation $A^\sim$ is intertwining for $U^{(2)}$ and $U^{(1)}$. In fact:

$$\begin{aligned}\langle U_x^{(1)} A^\sim \xi_2, \xi_1\rangle_1 &= \langle A^\sim \xi_2, U_{x^{-1}}^{(1)} \xi_1\rangle_1 = \langle \xi_2, A U_{x^{-1}}^{(1)} \xi_1\rangle_2 \\ &= \langle \xi_2, U_{x^{-1}}^{(2)} A\xi_1\rangle_2 = \langle U_x^{(2)}\xi_2, A\xi_1\rangle_2 = \langle A^\sim U_x^{(2)} \xi_2, \xi_1\rangle_1.\end{aligned}$$

Here ξ_j is arbitrary in H_j, so that $A^\sim U_x^{(2)} = U_x^{(1)} A^\sim$ for all $x \in G$.

Now we have $A^\sim A U_x^{(1)} = A^\sim U_x^{(2)} A = U_x^{(1)} A^\sim A$, *i.e.*, $A^\sim A$ commutes with all $U_x^{(1)}$. Clearly $A^\sim A \neq 0$, and we apply (21.30) to infer that $A^\sim A = \alpha I$ for some nonzero complex number α. For $\xi_1, \eta_1 \in H_1$, the foregoing yields

$$\alpha\langle \xi_1, \eta_1\rangle_1 = \langle A^\sim A\xi_1, \eta_1\rangle_1 = \langle A\xi_1, A\eta_1\rangle_2. \tag{1}$$

Let $\eta_1 = \xi_1 \neq 0$ in (1); this implies that α is real and positive. Let $\beta = \alpha^{-\frac{1}{2}}$; then (1) shows that βA is an isometry of H_1 onto H_2. □

(27.14) Theorem. *Notation is as in* (27.11). *Let $U^{(1)}$ and $U^{(2)}$ be irreducible. If there is a B for which the transformation A_B is nonzero, then $U^{(1)}$ and $U^{(2)}$ are equivalent representations.*

Proof. Both H_1 and H_2 are finite-dimensional. SCHUR's lemma (27.9) and (27.12) imply that A_B is a linear space isomorphism of H_1 onto H_2; (27.13) implies that βA_B is an intertwining linear isometry for a positive real number β. Thus $U^{(1)}$ and $U^{(2)}$ are equivalent. □

Theorem (27.14) contains the essentials of the orthogonality relations for functions in $\mathfrak{T}(G)$, which we now describe.

(27.15) Theorem. *Let σ_1 and σ_2 be distinct elements in Σ, and let f_j be a function in $\mathfrak{T}_{\sigma_j}(G)$ $(j = 1, 2)$. Then*

(i) $\int_G f_1 \bar{f}_2 \, d\lambda = 0$.

Proof. We select $U^{(1)}$ in σ_1 and $U^{(2)}$ in σ_2 and use the notation of (27.11). Choose orthonormal bases $\{\zeta_{j1}, \ldots, \zeta_{j d_j}\}$ in H_j $(j=1, 2)$. Linearity shows that (i) will be established as soon as we prove that

$$\int_G \langle U_x^{(1)} \zeta_{1j}, \zeta_{1k}\rangle_1 \overline{\langle U_x^{(2)} \zeta_{2l}, \zeta_{2m}\rangle_2}\, dx = 0 \tag{1}$$

for $j, k = 1, \ldots, d_1$ and $l, m = 1, \ldots, d_2$. Theorem (27.14) shows that the function $C_B(\xi_1, \xi_2)$ of (27.11.i) is zero for all $\xi_j \in H_j$, regardless of what linear transformation B is chosen. Fix indices $j, k \in \{1, 2, \ldots, d_1\}$ and $l, m \in \{1, 2, \ldots, d_2\}$. Let B be the linear transformation such that $B(\zeta_{1k}) = \zeta_{2m}$ and $B(\zeta_{1r}) = 0$ for $r \in \{1, 2, \ldots, k-1, k+1, \ldots, d_1\}$. We then have

$$\begin{aligned}
\langle B U_x^{(1)} \zeta_{1j}, U_x^{(2)} \zeta_{2l}\rangle_2 &= \left\langle \sum_{r=1}^{d_1} \langle U_x^{(1)} \zeta_{1j}, \zeta_{1r}\rangle_1 B\zeta_{1r}, \sum_{s=1}^{d_2} \langle U_x^{(2)} \zeta_{2l}, \zeta_{2s}\rangle_2 \zeta_{2s} \right\rangle_2 \\
&= \sum_{r=1}^{d_1} \sum_{s=1}^{d_2} \langle U_x^{(1)} \zeta_{1j}, \zeta_{1r}\rangle_1 \overline{\langle U_x^{(2)} \zeta_{2l}, \zeta_{2s}\rangle_2} \langle B\zeta_{1r}, \zeta_{2s}\rangle_2 \\
&= \sum_{r=1}^{d_1} \sum_{s=1}^{d_2} \langle U_x^{(1)} \zeta_{1j}, \zeta_{1r}\rangle_1 \overline{\langle U_x^{(2)} \zeta_{2l}, \zeta_{2s}\rangle_2}\, \delta_{rk}\, \delta_{ms} \\
&= \langle U_x^{(1)} \zeta_{1j}, \zeta_{1k}\rangle_1 \overline{\langle U_x^{(2)} \zeta_{2l}, \zeta_{2m}\rangle_2}.
\end{aligned}$$

Thus the integrand in (1) has the form of an integrand in (27.11.i), and as remarked above, this proves (1). ☐

In describing the orthogonality relations for equivalent representations, it is convenient to consider first a single representation.

(27.16) Lemma. *Let U be a continuous irreducible unitary representation of G with d-dimensional representation space H. Let B be any linear operator on H. Then the operator A_B defined in (27.11) is equal to $d^{-1}\operatorname{tr}(B)\cdot I$.*

Proof. The operator A_B is an intertwining operator for U: $A_B U_x = U_x A_B$ for all $x \in G$. Since U is irreducible, (21.30) or (27.10) implies that $A_B = \alpha_B I$, where $\alpha_B \in K$. Let $\{\zeta_1, \zeta_2, \ldots, \zeta_d\}$ be any orthonormal basis in H. Corollary (D.18) shows that

$$\operatorname{tr}(B) = \sum_{r=1}^{d} \langle B\zeta_r, \zeta_r\rangle.$$

For each $x \in G$, we have

$$\left.\begin{aligned}
\operatorname{tr}(B) &= \operatorname{tr}\left(U_x (U_{x^{-1}} B)\right) = \operatorname{tr}(U_{x^{-1}} B U_x) \\
&= \sum_{r=1}^{d} \langle U_{x^{-1}} B U_x \zeta_r, \zeta_r\rangle = \sum_{r=1}^{d} \langle B U_x \zeta_r, U_x \zeta_r\rangle.
\end{aligned}\right\} \tag{1}$$

Integrate (1) over G and apply (27.11) to obtain

$$\operatorname{tr}(B) = \sum_{r=1}^{d} \int_G \langle B U_x \zeta_r, U_x \zeta_r\rangle \, dx = \sum_{r=1}^{d} \langle A_B \zeta_r, \zeta_r\rangle = \sum_{r=1}^{d} \langle \alpha_B \zeta_r, \zeta_r\rangle = \alpha_B \, d.$$

Therefore we have $A_B = \alpha_B I = d^{-1} \operatorname{tr}(B) \cdot I$. □

(27.17) Theorem. *Let U, H, and d be as in (27.16). Let $\{\zeta_1, \zeta_2, \ldots, \zeta_t\}$ be an orthonormal set in H.*[1] *Then*

(i) $$\int_G \langle U_x \zeta_j, \zeta_k\rangle \overline{\langle U_x \zeta_l, \zeta_m\rangle} \, dx = \frac{1}{d} \delta_{jl} \delta_{km}$$

for $j, k, l, m \in \{1, 2, \ldots, t\}$. [Note that $\delta_{jl} \delta_{km} = 1$ if and only if the ordered pairs (j, k) and (l, m) are equal.]

Proof. Extend $\{\zeta_1, \zeta_2, \ldots, \zeta_t\}$ to an orthonormal basis $\{\zeta_1, \zeta_2, \ldots, \zeta_d\}$ in H. For $k, m \in \{1, 2, \ldots, d\}$, let B_{mk} be the linear operator on H such that $B_{mk}(\zeta_r) = \delta_{rk} \zeta_m$. Corollary (D.18) shows that $\operatorname{tr}(B_{mk}) = \delta_{km}$ and Lemma (27.16) thus shows that

$$A_{B_{mk}} = \frac{\delta_{km}}{d} I. \tag{1}$$

For $j, k, l, m \in \{1, 2, \ldots, d\}$, we have

$$\left.\begin{aligned} \langle B_{mk} U_x \zeta_j, U_x \zeta_l\rangle &= \left\langle \sum_{r=1}^{d} \langle U_x \zeta_j, \zeta_r\rangle B_{mk} \zeta_r, \sum_{s=1}^{d} \langle U_x \zeta_l, \zeta_s\rangle \zeta_s \right\rangle \\ &= \left\langle \sum_{r=1}^{d} \langle U_x \zeta_j, \zeta_r\rangle \delta_{rk} \zeta_m, \sum_{s=1}^{d} \langle U_x \zeta_l, \zeta_s\rangle \zeta_s \right\rangle \\ &= \left\langle \langle U_x \zeta_j, \zeta_k\rangle \zeta_m, \sum_{s=1}^{d} \langle U_x \zeta_l, \zeta_s\rangle \zeta_s \right\rangle \\ &= \sum_{s=1}^{d} \langle U_x \zeta_j, \zeta_k\rangle \overline{\langle U_x \zeta_l, \zeta_s\rangle} \langle \zeta_m, \zeta_s\rangle \\ &= \langle U_x \zeta_j, \zeta_k\rangle \overline{\langle U_x \zeta_l, \zeta_m\rangle}. \end{aligned}\right\} \tag{2}$$

Integrating both sides of (2) over G and applying (1) gives

$$\frac{\delta_{km} \delta_{jl}}{d} = \frac{\delta_{km}}{d} \langle \zeta_j, \zeta_l\rangle = \langle A_{B_{mk}} \zeta_j, \zeta_l\rangle = \int_G \langle U_x \zeta_j, \zeta_k\rangle \overline{\langle U_x \zeta_l, \zeta_m\rangle} \, dx. \quad □$$

[1] It is clear that $1 \leqq t \leqq d$; if $t = 1$, $\{\zeta_1\}$ consists of a single vector of length 1.