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A CHARACTERIZATION OF A CLASS OF $[Z]$ GROUPS VIA KOROVKIN THEORY

MANJU RANI AGRAWAL - U.B. TEWARI

We characterize all the central topological groups G for which the centre $Z(L^1(G))$ of the group algebra admits a finite universal Korovkin set. It is proved that $Z(L^1(G))$ has a finite universal Korovkin set iff $\hat{G}$ is a finite dimensional, separable metric space. This is equivalent to the fact that G is separable, metrizable and G/K has finite torsion free rank, where K is a compact open normal subgroup of certain direct summand of G.

1. Introduction.

Let A be a commutative Banach algebra with continuous involution. An eminent problem in Korovkin approximation Theory is to characterize those A which admit a finite universal Korovkin set. Here, a subset S of A is said to be a universal Korovkin set iff the following analogue of the classical Korovkin Theorem ([9]) is true:

For every commutative Banach algebra B with continuous symmetric involution, every *-homomorphism $T : A \to B$ and every uniformly bounded net $\{T_\alpha\}$ of positive linear operators from A to B, the convergence $\lim_\alpha \|(T_\alpha x - Tx)\hat{}\|_\infty = 0$ $\forall x \in S$ implies $\lim_\alpha \|(T_\alpha y - Ty)\hat{}\|_\infty = 0$ $\forall y \in A$.

In [1], we had characterized the central topological groups (or $[Z]$ groups) G having a compact open normal subgroup K such that $G =$

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KZ, where Z is the centre of G. for which the centre $Z(L^1(G))$ of the group algebra has a finite universal Korovkin set. In this connection it is appropriate to mention (see, [4, Theorem 4.4]) that every $[Z]$ group is of the form $G = V \times H$, where V is an Euclidean group and H has a compact open normal subgroup. Further, the group algebra of any Euclidean group has a finite universal Korovkin set. Therefore to prove that $Z(L^1(G))$ has a finite universal Korovkin set, it suffices to prove that $Z(L^1(H))$ has a finite universal Korovkin set, see [1]. Thus the basic problem is to investigate it for the $[Z]$ groups which have a compact open normal subgroup. In this paper we characterize such $[Z]$ groups for which $Z(L^1(G))$ has a finite universal Korovkin set.

If G is a $[Z]$ group having a compact open normal subgroup K. then we may assume, without loss of generality, that G/K is abelian [4, Cor. 2, p. 331]. Moreover, in this case there exists a finite chain of open normal subgroups of G such that

$$G = G_n \supseteq G_{n-1} \supseteq \cdots \supseteq G_1 \supseteq G_0 = KZ$$

and G_i/G_{i-1} is a cyclic group of prime order $\forall i = 1, \cdots, n$ (see, [7, Section 1.4, p. 70]).

In [1] we had already settled the problem for the case $n = 0$. In this paper we settle the problem for any $[Z]$ group which has a compact open normal subgroup and prove the following.

THEOREM 1.1. *Let G be a $[Z]$ gronp having a compact open normal subgroup K such that G/K is abelian. Then the following statements are equivalent.*

i) $Z(L^1(G))$ admits a finite universal Korovkin set.

ii) G is separable, metrizable and G/K has finite torsion free rank.

iii) $\hat{G}$ is a finite dimensional, separable metric space.

2. Notations and preliminaries.

We shall follow the notations used in [1]. A locally compact group G is said to be a $[Z]$ group if G/Z is compact, where Z is the centre of G. Throughout the paper G will be a $[Z]$ group. It is known [5, Theorem 2.1] that every continuons irreducible unitary representation of G is finite dimensional. $\hat{G}$ will denote the set of equivalence classes of continuous irreducible unitary representations of G. For $\sigma \in \hat{G}$, let χ_σ and d_σ be its character and dimension respectively and let τ_σ be the multiplicative linear functional on $Z(L^1(G))$ defined as

$$\tau_\sigma(f) = \int_G f(x) \frac{\overline{\chi_\sigma(x)}}{d_\sigma} dx, \ f \in Z(L^1(G)).$$

It is shown in [6, Section 6] that every multiplicative linear functional on $Z(L^1(G))$ arises in this manner and the set $\mathscr{X} = \left\{ \frac{\chi_\sigma}{d_\sigma} : \sigma \in \hat{G} \right\}$ equipped with the topology of uniform convergence on compact subsets of G coincides with the maximal ideal space of $Z(L^1(G))$. The topology of $\mathscr{X}$ can be transported to $\hat{G}$ in a natural way. Let H be a closed normal subgroup of G and $\sigma \in \hat{H}$ then σ^G will denote the representation of G induced by σ, see [8]. Let $S(\sigma) = \{s \in G : \sigma_{sxs^{-1}} = \sigma_x \forall x \in H\}$ denote the stability group of σ. For $\rho \in \hat{G}$, $\rho_{|H}$ will denote its restriction to H. By the dimension of a topological space we mean the covering dimension (see [10], p. 9).

3. Proof of the main result.

To prove Theorem 1.1, let us first collect some auxilliary results. Since each quotient G_i/G_{i-1} has a prime order, there is a nice relationship among $\hat{G}_i$'s. We quote the following useful Lemma in this connection.

LEMMA A [7, Lemma 1.1] *Let H be a normal subgroup (of prime index p) in a locally compact group G such that all the continuous irreducible unitary representations of G are finite dimensional. We define*

$$(\hat{G})_I = \{\rho \in \hat{G} : \rho_{|H} \text{ is irreducible}\},$$

$$(\hat{G})_{II} = \{\rho \in \hat{G} : \rho = \sigma^G \text{ for some } \sigma \in \hat{H}\},$$

$$(\hat{H})_I = \{\sigma \in \hat{H} : S(\sigma) = G\} \text{ and}$$

$$(\hat{H})_{II} = \{\sigma \in \hat{H} : S(\sigma) = H\}.$$

Then $\hat{G}$ is the disjoint union of $(\hat{G})_I$ and $(\hat{G})_{II}$; $\hat{H}$ is the disjoint union of $(\hat{H})_I$ and $(\hat{H})_{II}$. Moreover we have

(i) If $\sigma \in \hat{H}$, then $\sigma \in (\hat{H})_I$ iff $\sigma = \rho_{|H}$ for some $\rho \in (\hat{G})_I$. In this case all the extentions of σ are of the form $\tilde{\chi} \otimes \rho$ for $\chi \in (G/H)\hat{}$, where $\tilde{\chi}$ denotes the lift of χ to G.

(ii) If $\sigma \in \hat{H}$, then $\sigma \in (\hat{H})_{II}$ iff σ^G is irreducible. Further, for $\rho \in \hat{G}$, $\rho \in (\hat{G})_{II}$ iff $\chi_\rho(t) = 0$ for $t \notin H$.

We shall also need the following.

LEMMA 3.1. *Let H and G be as in Lemma A. If $\rho \in \hat{G}$ and $\rho_{|H}$ is irreducible then $\chi_\rho(x) \neq 0$ $\forall x \in G$.*

Proof. By (i) of Lemma A, $\rho_{|H} \in (\hat{H})_I$ and therefore the stability group of $\rho_{|H}$ is G. Thus we have

$$\begin{aligned} G &= \{s \in G : (\rho_{|H})_{sxs^{-1}} = (\rho_{|H})_x \forall x \in H\} \\ &= \{s \in G : \rho_{sxs^{-1}} = \rho_x \forall x \in H\} \\ &= \{s \in G : \rho_s \rho_x \rho_{s^{-1}} = \rho_x \forall x \in H\} \\ &= \{s \in G : \rho_s \rho_x = \rho_x \rho_s \forall x \in H\}. \end{aligned}$$

Since $\{\rho_x : x \in H\}$ is an irreducible set of operators, it follows that $\rho_s = c_\rho(s) I_\rho$, where I_ρ is the d_ρ-dimensional identity operator and $c_\rho(s)$ is a scalar depending on ρ and s. It is easy to check that $c_\rho(st) = c_\rho(s)c_\rho(t)$ for $s, t \in G$ and $c_\rho(e) = 1$. Since $\chi_\rho(x) = c_\rho(x) d_\rho$, χ_ρ can not vanish on G.

(3.2) Proof of Theorem 1.1

(i)⇒(iii). It has been shown in [1] that if $Z(L^1(G))$ has a finite universal Korovkin set then its maximal ideal space $\mathscr{X}$ is a finite dimensional separable metric space. Consequently $\hat{G}$ is also a finite dimensional separable metric space.

(iii)⇒(ii). Since G/K is abelian, K contains the closure of the commutator group G'. Now as in the proof of [1, Theorem 4.2], it can be shown that G/K has finite torsion free rank. Further, by Theorem 2.3 of [7], G is separable and metrizable.

(ii)$\Rightarrow$(i). Since $Z(L^1(G))$ has a bounded approximate identity, in view of Theorem 2.9 of [1] it suffices to show that there exist finitely many functions in $Z(L^1(G))$ such that their Gelfand transforms separate the points of $\mathscr{X}$. As discussed in Section 1, there exists a finite chain of open normal subgroups of G such that

$$G = G_n \supseteq G_{n-1} \supseteq \cdots \supseteq G_1 \supseteq G_0 = KZ$$

and G_i/G_{i-1} is a cyclic group of prime order $\forall i = 1, \ldots, n$. Further, since G/K has finite torsion free rank, each G_i/K $(i = 0, ..., n)$ has finite torsion free rank. We shall prove the assertion by induction on the length of the normal series. The case $n = 0$ has already been taken care of [1, Theorem 4.2].

Step 1.

We assume that $n = 1$, that is $G = G_1 \supseteq G_0 = KZ$ and G/G_0 is a cyclic group of prime order p. Let y be an arbitrary but fixed element in G such that the coset yG_0 is a generator of G/G_0. Since G is separable and metrizable, so is K. Hence $\hat{K}$ is countable. Since K is a compact subgroup of G, by Theorem 5.1 of [5], $\hat{K} \subseteq \{\sigma_{|K} : \sigma \in \hat{G}\}$. Thus we may choose a sequence $\{\sigma_n\}_{n=1}^{\infty}$ in $\hat{G}$ such that $\hat{K} = \{\sigma_{n|K}\}_{n=1}^{\infty}$. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence of distinct positive numbers such that

$$\sum_{n=1}^{\infty} \alpha_n d_{\sigma_n}^2 < \infty.$$

Now G_0/K has finite torsion free rank, say r. Then there exist $x_1, \ldots, x_r \in G_0$ such that $\{x_1K, ..., x_rK\}$ is a maximal independent set of elements of infinite order in G_0/K. Since $G_0 = KZ$, $x_1, \ldots, x_r$ may be chosen to be central elements of G. Let H be the subgroup generated by K and $x_1, \ldots, x_r$. Then H is an open normal subgroup of G_0 and G_0/H is discrete. Since G is separable and metrizable, G_0/H is second countable. Therefore G_0/H is countable. Further, since $G_0 = HZ$, we may choose elements $\{s_i\}_{i=1}^{\infty}$ belonging to Z such that $G_0/H = \{s_iH\}_{i=1}^{\infty}$. Since G_0/H is a torsion group, $\{\gamma(s_iH) : \gamma \in (G_0/H)\hat{}\}$ is a finite subgroup of the circle group for each i. Therefore for each i, there exists ε_i, $0 < \varepsilon_i < 1$ and such that

$$|\gamma(s_iH) - 1| \geq \varepsilon_i \forall \gamma \in (G_0/H)\hat{} \text{ for which } \gamma(s_iH) \neq 1.$$

Let $\{\delta_i\}_{i=1}^{\infty}$ be a summable sequence of positive numbers satisfying

$$\varepsilon_n \delta_n - 2 \sum_{i \geq n+1} \delta_i > 0 \; \forall n.$$

(one such sequence is obtained by defining $\delta_i = \dfrac{1}{4i} \prod_{k=1}^{i-1} \varepsilon_k$, as in [11, Section 3, p. 453]).

We define $r+3$ functions in $Z(L^1(G))$ as follows:

$$f_0 = \sum_{n=1}^{\infty} \alpha_n d_{\sigma_n} \chi_{\sigma_n} \xi_K$$

$$f_i(x) = f_0(x x_i^{-1}) \xi_{G_0}, \; 1 \leq i \leq r$$

$$f_{r+1}(x) = \sum_{i=1}^{\infty} \delta_i f_0(x s_i^{-1}) \xi_{G_0}$$

$$f_{r+2}(x) = f_0(x y^{-1}),$$

where for a subset E of G, ξ_E denotes the characteristic function of E.

We shall prove that the Gelfand transforms of these $r+3$ functions separate the points of $\mathscr{X}$. Note that the Gelfand transforms of the restrictions of the functions $f_0, \ldots, f_{r+1}$ to G_0 already separate the points of $\mathscr{X}_{G_0} = \left\{ \dfrac{\chi_\sigma}{d_\sigma} : \sigma \in \hat{G}_0 \right\}$ (cf. Theorem 4.2 of [1]). Let $\sigma, \rho \in \hat{G}$ satisfy $\tau_\sigma(f_i) = \tau_\rho(f_i) \forall i = 0, 1, \ldots, r+2$. We must show that $\sigma = \rho$.

In view of Lemma A, there are three possibilities:

(i) $\sigma \in (\hat{G})_I, \; \rho \in (\hat{G})_{II}$.

(ii) $\sigma, \rho \in (\hat{G})_I$,

(iii) $\sigma, \rho \in (\hat{G})_{II}$.

Case (i) $\sigma \in (\hat{G})_I, \; \rho \in (\hat{G})_{II}$.

We shall show that this possibility can not occur. The assumption implies that $\sigma_{|G_0}$ is irreducible and $\rho = \eta^G$ (induced representation) for some $\eta \in \hat{G}_0$. Now

$$\tau_\sigma(f) = \frac{1}{d_\sigma} \int_G f(x) \overline{\chi_\sigma(x)} dx$$

and

$$\begin{aligned}\tau_\rho(f) &= \frac{1}{d_\rho}\int_G f(x)\overline{\chi_\rho(x)}dx \\ &= \frac{1}{pd_\eta}\int_{G_0} f(x)p\overline{\chi_\eta(x)}dx = \frac{1}{d_\eta}\int_{G_0} f(x)\overline{\chi_\eta(x)}dx.\end{aligned}$$

Thus

$$\begin{aligned}\tau_\sigma(f_i) = \tau_\rho(f_i) \quad \forall i = 0, \ldots, r+1 \\ \Rightarrow \tau_{\sigma|G_0}(f_i) = \tau_\eta(f_i) \quad \forall i = 0, \ldots, r+1 \\ \Rightarrow \sigma_{|G_0} = \eta.\end{aligned}$$

But this is a contradiction because

$$\sigma_{|G_0} \in (\hat{G}_0)_I = \{\sigma \in \hat{G}_0 : S(\sigma) = G\}$$

and $\eta \in (\hat{G}_0)_{II} = \{\sigma \in \hat{G}_0 : S(\sigma) = G_0\}$ and the two sets are disjoint (see, Lemma A).

Case (ii) $\sigma, \rho \in (\hat{G})_I$.

The assumption implies that $\sigma_{|G_0}$ and $\rho_{|G_0}$ are irreducible. Thus the equations $\tau_\sigma(f_i) = \tau_\rho(f_i)\forall i = 0, \ldots, r+1$ imply that $\sigma_{|G_0} = \rho_{|G_0}$. It follows from Lemma A (i) that

$$\sigma = \tilde{\chi} \otimes \rho,$$

where $\chi \in (G/G_0)\hat{}$ and $\tilde{\chi}$ is the lift of χ to G. We claim that $\tilde{\chi} \equiv 1$.

Now $\sigma_{|G_0}$ and $\rho_{|G_0}$ are irreducible. Since $G_0 = KZ$, by Lemma 4.1 of [1], it follows that $\sigma_{|K}$ and $\rho_{|K}$ are irreducible. If $\sigma_{|K} = \sigma_{m|K} = \rho_{|K}$ then

$$\begin{aligned}\tau_\sigma(f_{r+2}) &= \frac{1}{d_\sigma}\int_G \sum_{n=1}^{\infty} \alpha_n d_{\sigma_n} \chi_{\sigma_n}(xy^{-1})\xi_K(xy^{-1})\overline{\chi_\sigma(x)}dx \\ &= \frac{1}{d_\sigma}\int_K \sum_{n=1}^{\infty} \alpha_n d_{\sigma_n} \chi_{\sigma_n}(x)\overline{\chi_\sigma(xy)}dx = \frac{\alpha_m}{d_{\sigma_m}}\overline{\chi_\sigma(y)}\end{aligned}$$

as can be easily proved. Similarly

$$\tau_\rho(f_{r+2}) = \frac{\alpha_m}{d_{\sigma_m}}\overline{\chi_\rho(y)}$$

Thus

$$\begin{aligned} \tau_\sigma(f_{r+2}) &= \tau_\rho(f_{r+2}) \\ \Rightarrow \chi_\sigma(y) &= \chi_\rho(y) \\ \Rightarrow \chi_{\tilde{\chi}\otimes\rho} &= \chi_\rho(y) \\ \Rightarrow \tilde{\chi}(y)\chi_\rho(y) &= \chi_\rho(y) \\ \Rightarrow \tilde{\chi}(y) &= 1, \end{aligned}$$

since by Lemma 3.1 $\chi_\rho(y) \neq 0$. Therefore $\chi(yG_0) = 1$. Since yG_0 is a generator of the cyclic group G/G_0, it follows that $\tilde{\chi} \equiv 1$. Consequently $\sigma = \rho$.

Case (iii) $\sigma, \rho \in (\hat{G})_{II}$.

The assumption implies that there exist $\mu, \eta \in \hat{G}_0$ such that $\sigma = \mu^G$ and $\rho = \eta^G$. Now for $f \in Z(L^1(G))$,

$$\begin{aligned} \tau_\sigma(f) &= \frac{1}{d_\sigma}\int_G f(x)\overline{\chi_\sigma(x)}dx \\ &= \frac{1}{pd_\mu}\int_{G_0} f(x)p\overline{\chi_\mu(x)}dx = \frac{1}{d_\mu}\int_{G_0} f(x)\overline{\chi_\mu(x)}dx \end{aligned}$$

and similarly

$$\tau_\rho(f) = \frac{1}{d_\eta}\int_{G_0} f(x)\overline{\chi_\eta(x)}dx.$$

Thus the equations $\tau_\sigma(f_i) = \tau_\rho(f_i)$ $\forall i = 0, ..., r+1$ yield that $\mu = \eta$. Since σ and ρ are respectively the representations on G induced by μ and η, we have $\sigma = \rho$.

Step 2.

Assuming that the assertion is true for $Z(L^1(G_{j-1}))$ we shall establish the assertion for $Z(L^1(G_j))$. Let p_i be the prime order of the cyclic group G_i/G_{i-1}. For each $i = 1, \ldots, j$, we fix $y_i \in G_i$ such that y_iG_{i-1} is a generator of the cyclic group G_i/G_{i-1}. Since K is a compact subgroup of G_j and $\hat{K}$ is countable, as in Step 1, we may choose a sequence $\{\sigma_n\}_{n=1}^{\infty}$ in $\hat{G}_j$ such that $\hat{K} = \{\sigma_{n|K}\}_{n=1}^{\infty}$. Note that the restriction of each σ_n to each G_i, $0 \le i \le j-1$, is irreducible.

Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence of distinct positive numbers such that

$$\sum_{n=1}^{\infty} \alpha_n d_{\sigma_n}^2 < \infty.$$

Now G_0/K has finite torsion free rank, say r. Let $x_1, \ldots, x_r$, $\{s_i\}_{i=1}^{\infty}$, $\{\varepsilon_i\}_{i=1}^{\infty}$ and $\{\delta_i\}_{i=1}^{\infty}$ be as in Step 1. We define $r+j+2$ functions in $Z(L^1(G_j))$ as follows:

$$\begin{aligned}
f_0 &= \sum_{n=1}^{\infty} \alpha_n d_{\sigma_n} \chi_{\sigma_n} \xi_K \\
f_i(x) &= f_0(x x_i^{-1}) \xi_{G_{j-1}}, \quad 1 \le i \le r \\
f_{r+1}(x) &= \sum_{i=1}^{\infty} \delta_i f_0(x s_i^{-1}) \xi_{G_{j-1}} \\
f_{r+2}(x) &= f_0(x y_1^{-1}) \xi_{G_{j-1}} \\
f_{r+3}(x) &= f_0(x y_2^{-1}) \xi_{G_{j-1}} \\
&\vdots \\
f_{r+j}(x) &= f_0(x y_{j-1}^{-1}) \xi_{G_{j-1}} \\
f_{r+j+1}(x) &= f_0(x y_j^{-1})
\end{aligned}$$

Note that the restrictions of the functions $f_0, \ldots, f_{r+j}$ to G_{j-1} are in $Z(L^1(G_{j-1}))$ and their Gelfand transforms separate the points of $\mathscr{X}_{G_{j-1}} = \left\{ \frac{\chi_\sigma}{d_\sigma} : \sigma \in \hat{G}_{j-1} \right\}$. We claim that $\hat{f}_0, \ldots, \hat{f}_{r+j+1}$ separate the points of $\mathscr{X}_{G_j} = \left\{ \frac{\chi_\sigma}{d_\sigma} : \sigma \in \hat{G}_j \right\}$. Let $\sigma, \rho \in \hat{G}_j$ satisfy $\tau_\sigma(f_i) = \tau_\rho(f_i)$ $\forall i = 0, 1, \ldots, r+j+1$. Since G_j/G_{j-1} has prime order, as before there are three possibilities:

(i') $\sigma \in (\hat{G}_j)_I$ and $\rho \in (\hat{G}_j)_{II}$

(ii') $\sigma, \rho \in (\hat{G}_j)_I$

(iii') $\sigma, \rho \in (G_j)_{II}$,

Case (i′)

$\sigma \in (\hat{G}_j)_I$ and $\rho \in (\hat{G}_j)_{II}$ that is $\sigma_{|G_{j-1}}$ is irreducible and $\rho = \eta^{G_j}$, for some $\eta \in \hat{G}_{j-1}$. Now as in Step 1,

$$\tau_\sigma(f) = \frac{1}{d_\sigma}\int_{G_j} f(x)\overline{\chi_\sigma(x)}dx$$

and

$$\tau_\rho(f) = \frac{1}{d_\eta}\int_{G_{j-1}} f(x)\overline{\chi_\eta(x)}dx.$$

Thus

$$\tau_\sigma(f_i) = \tau_\rho(f_i)\ \forall i = 0, \dots, r+j$$

$\Rightarrow \tau_{\sigma_{|G_{j-1}}}(f_i) = \tau_\eta(f_i) \quad \forall i = 0, \dots, r+j$

$\Rightarrow \sigma_{|G_{j-1}} = \eta$. This leads to a contradiction as in Step 1.

Case (ii′)

$\sigma, \rho \in (\hat{G}_j)_I$ that is $\sigma_{|G_{j-1}}$ and $\rho_{|G_{j-1}}$, are irreducible. The equatios $\tau_\sigma(f_i) = \tau_\rho(f_i)\ \forall i = 0, \dots, r+j$ imply that $\sigma_{|G_{j-1}} = \rho_{|G_{j-1}}$. Therefore, $\sigma = \tilde{\chi} \otimes \rho$, where $\tilde{\chi}$ is the lift of $\chi \in (G_j/G_{j-1})\hat{}$ to G_j. Again as in Step 1, we can conclude from the equation $\tau_\sigma(f_{r+j+1}) = \tau_\rho(f_{r+j+1})$ that $\tilde{\chi} \equiv 1$. Thus $\sigma = \rho$.

Case (iii′) that is $\sigma = \mu^{G_j}$ and $\rho = \eta^{G_j}$, where $\mu, \eta \in \hat{G}_{j-1}$. As in step 1,

$$\begin{aligned}\tau_\sigma(f) &= \frac{1}{d_\sigma}\int_{G_j} f(x)\overline{\chi_\sigma(x)}dx \\ &= \frac{1}{p_j d_\mu}\int_{G_{j-1}} f(x)p_j\overline{\chi_\mu(x)}dx = \frac{1}{d_\mu}\int_{G_{j-1}} f(x)\overline{\chi_\mu(x)}dx.\end{aligned}$$

and

$$\tau_\rho(f) = \frac{1}{d_\eta}\int_{G_{j-1}} f(x)\overline{\chi_\eta(x)}dx.$$

The equations $\tau_\sigma(f_i) = \tau_\rho(f_i)\ \forall i = 0, \dots, r+j$ now imply that $\mu = \eta$ and hence $\sigma = \rho$.

This completes the proof.