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4-Dimensional Elation Laguerre Planes Admitting Non-Solvable Automorphism Groups

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Abstract. This paper concerns 4-dimensional (topological locally compact connected) elation Laguerre planes that admit non-solvable automorphism groups. It is shown that such a plane is either semi-classical or a single plane admitting the group SL(2, $\mathbb{R}$). Various characterizations of this single Laguerre plane are obtained.

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1. Introduction

A *4-dimensional Laguerre plane* $\mathscr{L} = (P, \mathscr{C}, \|)$ is an incidence structure consisting of a 4-dimensional locally compact point set P, a circle set $\mathscr{C}$ homeomorphic to $\mathbb{R}^6$ whose elements are subsets of P, called circles, and an equivalence relation $\|$ (parallelism) defined on the point set such that three mutually non-parallel points can be joined by a unique circle, such that the circles which touch a fixed circle K at $p \in K$ partition $P \setminus |p|$ (where $|p|$ denotes the parallel class of p), such that each parallel class meets each circle in a unique point (parallel projection), and such that there is a circle that contains at least three points. Furthermore, the operations of joining three points by a circle, of intersecting two different circles, and touching are continuous with respect to the induced topologies on their respective domains of definition. For more information about topological Laguerre planes we refer to [6], [7] and [19].

Circles and parallel classes of 4-dimensional Laguerre planes are homeomorphic to the 2-sphere $\mathbb{S}_2$ and to $\mathbb{R}^2$, respectively. Furthermore, the space Π of all parallel classes can be identified with a circle and thus is homeomorphic to $\mathbb{S}_2$. The point set of a 4-dimensional Laguerre plane can be considered as a fibre bundle over a circle with fibres being the parallel classes. The projection map of the bundle is given by the parallel projection in the Laguerre plane. It was shown in [18] that the point space of a 4-dimensional Laguerre plane is bundle-equivalent to the tangent bundle of the 2-sphere $\mathbb{S}_2$. Furthermore, in a 4-dimensional Laguerre plane any two circles have at least one point in common, cf. [7, Satz 3.3.b] and [3, Satz 3.7.b].

The *classical* (4-*dimensional*) *complex Laguerre plane* is obtained as the geometry of non-trivial plane sections of an elliptic cone in complex projective 3-dimensional space with its vertex removed.

In [16], we constructed the first examples of non-classical 4-dimensional Laguerre planes. They were obtained by pasting together two halves of the classical complex Laguerre plane along a 3-dimensional separating set in the point set and are therefore called *semi-classical* planes; see 2.5 for an explicit description. These planes were characterized in [20] in terms of their automorphism groups and how they act on the sets of parallel classes and on the circle sets; see Theorem 2.5. In this paper we continue to study 4-dimensional Laguerre planes and determine all such planes admitting non-solvable automorphism groups and a full elation group, see below for a definition of elation group.

2. Four-Dimensional Elation Laguerre Planes

The collection of all continuous automorphisms of a 4-dimensional Laguerre plane $\mathscr{L}$ forms a group with respect to composition, the automorphism group Γ of $\mathscr{L}$. This group is a Lie group with respect to the compact-open topology, see [4, Satz 3.9] or [14]. The *kernel* of a Laguerre plane consists of all automorphisms that fix each parallel class. This collection of automorphisms is a closed normal subgroup of the automorphism group of the Laguerre plane.

The maximum dimensions of the automorphism group and kernel of a 4-dimensional Laguerre plane are 14 and 8, respectively. These dimensions are attained in the classical complex Laguerre plane: The kernel is isomorphic to $\mathbb{C}\backslash\{0\} \ltimes \mathbb{C}^3$ and the connected component of the automorphism group of the classical 4-dimensional Laguerre plane that contains the identity is isomorphic to $(\mathbb{C}\backslash\{0\} \times \mathrm{SO}_3(\mathbb{C})) \ltimes \mathbb{C}^3$ (all collineations of the ambient complex projective 3-dimensional space that leave the elliptic cone and its vertex invariant).

Every 4-dimensional Laguerre plane contains one further distinguished closed normal subgroup, the *elation group* Δ, which is the collection of all automorphisms in the kernel T that fix no circle plus the identity. We obtain an *elation Laguerre plane* if Δ acts sharply transitively on the set of circles, see [17]. The automorphisms in this group induce elations in the associated Lie geometry and one obtains an elation generalized quadrangle; cf. [13] for generalized quadrangles and their relation to Laguerre planes and the other types of circle planes. The following characterizations of 4-dimensional elation Laguerre planes were obtained in [17], see also [10].

Theorem 2.1. *For a* 4-*dimensional Laguerre plane* $\mathscr{L}$ *let* Δ *and* T *be the elation group and the kernel of* $\mathscr{L}$, *respectively. Then the following are equivalent*:

(1) $\mathscr{L}$ *is an elation Laguerre plane*;
(2) *the kernel* T *is transitive on the set of circles*;
(3) T *is at least* 7-*dimensional*;
(4) Δ *is* 6-*dimensional*;
(5) Δ *is connected and isomorphic to* $\mathbb{R}^6$;
(6) Δ *is sharply transitive on the set of circles*.

Associated with every point p of a 4-dimensional Laguerre plane $\mathscr{L}$ there is a derived incidence structure, called the *derived affine plane* $\mathscr{A}_p = (A_p, \mathscr{L}_p)$ at p, whose point set $A_p \approx \mathbb{R}^4$ consists of all points of $\mathscr{L}$ that are not parallel to p and whose line set $\mathscr{L}_p$ consists of all restrictions to A_p of circles of $\mathscr{L}$ passing through p and of all parallel classes not passing through p. From the axioms of a Laguerre plane it readily follows that $\mathscr{A}_p$ is an affine plane. Indeed, each derived affine plane $\mathscr{A}_p$ of a 4-dimensional Laguerre plane is even a topological locally compact connected affine plane; see the coherence axioms in [6] and [7]. Moreover, $\mathscr{A}_p$ extends to a topological locally compact connected projective plane $\mathscr{P}_p$, which we call the *derived projective plane at p*; compare [12, Corollary 43.7].

The axioms of a 4-dimensional Laguerre plane further imply that circles not passing through the distinguished point p induce ovals in $\mathscr{P}_p$ by removing the point parallel to p and adding in $\mathscr{P}_p$ the infinite point of the lines that come from parallel classes of $\mathscr{L}$. The line at infinity of $\mathscr{P}_p$ (relative to $\mathscr{A}_p$) is a tangent to each of these ovals. We even obtain closed ovals in this way; compare [12, Proposition 55.18 and Theorem 55.11]. In $\mathscr{A}_p$ one has a *parabolic curve*. This gives a very convenient description of a Laguerre plane in one derived affine plane. One has the lines of the affine plane and a collection of parabolic curves. To obtain the entire Laguerre plane, however, one has to extend this model by one parallel class.

The classical complex Laguerre plane can be characterized in various ways, cf. [4], [9], [15] and [20]. We give a summary of these characterizations.

Theorem 2.2. *A* 4*-dimensional Laguerre plane* $\mathscr{L}$ *is isomorphic to the classical complex Laguerre plane if and only if at least one of the following holds*:

(1) *one derived affine plane is Desarguesian*;
(2) *the automorphism group of* $\mathscr{L}$ *is at least* 11*-dimensional*;
(3) *the kernel T of* $\mathscr{L}$ *is* 8*-dimensional.*

Assuming that $\mathscr{L}$ *is even an elation Laguerre plane with automorphism group* Γ, *then the classical complex Laguerre plane can be further characterized by each of the following*:

(4) Γ *is flag-transitive*;
(5) Γ *is transitive on the points of* $\mathscr{L}$;
(6) Γ *is transitive on the set of parallel classes of* $\mathscr{L}$;
(7) Γ *contains a compact group of dimension at least* 2.

The beauty of elation Laguerre planes lies in the fact that each derived projective plane of an elation Laguerre plane is a dual translation plane with centre ω (the infinite point of lines that come from parallel classes of the Laguerre plane). The stabilizer of a circle is linearly represented on the elation group Δ by conjugation. Therefore the well developed theory of translation planes, see [12], and, if the stabilizer of a circle is large enough, the representation theory of Lie groups can be applied to classify the most homogeneous elation Laguerre planes. This was carried out in [15] and all 4-dimensional elation Laguerre planes that admit 4-dimensional stabilizers of a circle were classified. We shall see that there is, up to isomorphism, a unique 4-dimensional elation Laguerre plane that admits an almost simple but non-simple Lie group of automorphisms.

2.3. A matrix representation of 4-dimensional elation Laguerre planes. For each $z \in \mathbb{S}_2 \approx \mathbb{R}^2 \cup \{\infty\}$ let

$$D(z) = \begin{pmatrix} A(z) \\ B(z) \\ C(z) \end{pmatrix}$$

be a 6×2 matrix with 2×2 matrices A, B, C such that $A(\infty) = I_2$ and $B(\infty) = C(\infty) = 0$ where 0 and I_2 denote the 2×2 zero and identity matrix, respectively, and such that for all $z \in \mathbb{R}^2$ the second row of $B(z)$ equals z and $C(z) = I_2$. One can further assume that $A(0) = B(0) = 0$. Every 4-dimensional elation Laguerre plane can be represented by such a mapping D.

Circles of the elation Laguerre plane $\mathscr{L}$ are of the form

$$K_c = \{(z, c \cdot D(z)) | z \in \mathbb{S}_2\}$$

where $c \in \mathbb{R}^6$. The elation group Δ of $\mathscr{L}$ is given by all maps

$$(z, w) \mapsto (z, w + c \cdot D(z))$$

for $c \in \mathbb{R}^6$; the connected component of the kernel containing the identity consists of all maps

$$(z, w) \mapsto (z, rw + c \cdot D(z))$$

for $c \in \mathbb{R}^6$, $r \in \mathbb{R}$, $r > 0$. The continuity of the geometric operations in such a Laguerre plane described by a matrix valued mapping D reduces to the continuity of D in $\mathbb{R}^2$ and to $\lim_{z \to \infty} D(z) A(z)^{-1} = D(\infty)$, cf. [17, Proposition 5.8].

Note that a circle K_c passes through the infinite point $(\infty, (c_1, c_2))$ where c_1 and c_2 denote the first and second component of the vector $c \in \mathbb{R}^6$. Hence, circles through the point $(\infty, 0)$ are of the form

$$\{(z, (c_3, c_4) B(z) + (c_5, c_6)) | z \in \mathbb{S}_2\}$$

for $c_3, c_4, c_5, c_6 \in \mathbb{R}$. Since this derived affine plane is a dual translation plane, one finds that $\{B(z) | z \in \mathbb{R}^2\}$ is a spread set.

It further follows that the circles that touch K_0 at $(\infty, 0)$ are of the form $\{(z, (c_5, c_6)) | z \in \mathbb{S}_2\}$ for $c_5, c_6 \in \mathbb{R}$. In particular, the bundle of all circles through $(\infty, 0)$ yields a 4-dimensional vector subspace of $\mathbb{R}^6$ and the bundle of all circles that touch K_0 at $(\infty, 0)$ yields a 2-dimensional vector subspace of $\mathbb{R}^6$. This carries over to arbitrary bundles; compare [17, 4.5].

Also note that

$$\{\{(z, (c_3, c_4) B(z)) | z \in \mathbb{S}_2\} | c_3, c_4 \in \mathbb{R}\}$$

and

$$\{\{(z, (c_1, c_2) A(z)) | z \in \mathbb{S}_2\} | c_1, c_2 \in \mathbb{R}\}$$

represent the bundle of circles through $(\infty, 0)$ and $(0, 0)$ and the bundle of circles that touch K_0 at $(0, 0)$, respectively.

2.4. Example. The semi-classical Laguerre planes from [16] can be described in the above fashion as follows. For $0 < p \leqslant 1$ and $x, y \in \mathbb{R}$ let

$$A(x,y) = \begin{pmatrix} y^2 - x^2 & -2xy \\ 2xy & y^2 - x^2 \end{pmatrix}, \quad B(x,y) = \begin{pmatrix} y & -x \\ x & y \end{pmatrix}, \quad \text{for } y \geqslant 0,$$
$$A(x,y) = \begin{pmatrix} py^2 - x^2 & -2xy \\ 2pxy & py^2 - x^2 \end{pmatrix}, \quad B(x,y) = \begin{pmatrix} py & -x \\ x & y \end{pmatrix}, \quad \text{for } y \leqslant 0.$$

Every semi-classical Laguerre plane is isomorphic to an elation Laguerre plane as described above for exactly one p, $0 < p \leqslant 1$. One obtains the classical complex Laguerre plane for $p = 1$.

The automorphism group of $\mathscr{L}_p$ is 10-dimensional unless $p = 1$. In any case, a semi-classical Laguerre plane admits the simple group $\mathrm{PSL}_2(\mathbb{R})$ as a group of automorphisms. In fact, the semi-classical Laguerre planes can be characterized among 4-dimensional elation Laguerre planes by this property, see [20, Theorem 5.8].

Theorem 2.5. *A* 4*-dimensional elation Laguerre plane is semi-classical if and only if it admits a simple Lie group of automorphisms.*

3. The $\mathrm{SL}_2(\mathbb{R})$-Elation Laguerre Plane

In this section $\mathscr{L} = (P, \mathscr{C}, \|)$ always denotes a 4-dimensional elation Laguerre plane and Γ, T and $\Delta \cong \mathbb{R}^6$ denote its automorphism group, kernel and elation group, respectively. For a subgroup Φ of Γ we denote by Φ^1 the connected component of Φ containing the identity. We say that a subgroup Σ of Γ is in the *elation complement* if its intersection with Δ is trivial, that is, $\Sigma \cap \Delta = \{\mathrm{id}\}$. Since we can replace a subgroup in the elation complement by an isomorphic subgroup in the elation complement that fixes a circle, cf. [20, Lemma 4.1], we only consider subgroups in the elation complement that fix a circle.

A subgroup Σ in the stabilizer of a circle K operates on the circle set $\mathscr{C}$ and on the elation group Δ by conjugation. The operation on Δ yields a linear and faithful representation of Σ on $\mathbb{R}^6$. Furthermore, $\mathscr{C}$ and Δ can be identified in such a way that the distinguished circle K corresponds to the identity. More precisely, $\delta \in \Delta$ is identified with $\delta(K)$. Then both actions are equivalent, that is, $\sigma\delta\sigma^{-1} \in \Delta$ is identified with $\sigma\delta(K)$ for $\sigma \in \Sigma$. In the matrix representation 2.4 of an elation Laguerre plane a circle L has coordinate vector c and this vector c also describes the corresponding element of Δ. We always make this identification.

Suppose that the automorphism group Γ of a 4-dimensional elation Laguerre plane $\mathscr{L}$ is 10-dimensional. Let K be a circle of $\mathscr{L}$ and let Σ be the connected component of the group $\{\gamma \in \Gamma_K | \det(\gamma) = 1\}$ that contains the identity. Then Σ is a closed, connected 3-dimensional subgroup of Γ_K^1 and has discrete intersection with T. Furthermore, Γ_K^1 is the semidirect product of Σ and $(\Gamma_K \cap T)^1$ and Σ is locally isomorphic to $\Gamma^1/(\Gamma^1 \cap T)$.

A non-solvable Lie group contains a maximal semisimple subgroup (Levi complement). Furthermore, a semisimple Lie group is locally isomorphic to a direct product of almost simple (or locally simple) Lie groups, cf. [5, Theorem 19.12] or

[12, 94.23]. We begin with a classification of semisimple Lie groups that can occur as groups of automorphisms of elation Laguerre planes.

Proposition 3.1. *Let $\mathscr{L}$ be a 4-dimensional elation Laguerre plane and assume that $\mathscr{L}$ admits a connected semisimple Lie group Σ of automorphisms. Then Σ is in the elation complement and has discrete intersection with the kernel T. Furthermore, Σ is almost simple and isomorphic to* $\mathrm{PSL}_2(\mathbb{C})$, $\mathrm{SO}_3(\mathbb{R})$, $\mathrm{PSL}_2(\mathbb{R})$ *or* $\mathrm{SL}_2(\mathbb{R})$.

Proof. Let Σ be a semisimple Lie group of automorphisms of $\mathscr{L}$. Since Δ is an abelian normal subgroup of Γ, we clearly see that Σ must be in the elation complement. Likewise, the connected component T^1 of the kernel T is a solvable normal subgroup of Γ, see [17, Corollary 3.6], so that Σ must have discrete intersection with T.

A semisimple Lie group is at least 3-dimensional and, because $\dim T \geqslant 7$, the dimension of Γ is at least 10. Hence either $\mathscr{L}$ is classical or $\dim \Sigma = 3$ by Theorem 2.2. In the former case, Σ is a subgroup of $\mathrm{PSL}_2(\mathbb{C})$ and thus isomorphic to $\mathrm{PSL}_2(\mathbb{C})$, $\mathrm{SO}_3(\mathbb{R})$ or $\mathrm{PSL}_2(\mathbb{R})$. In the latter case, Σ is almost simple and locally isomorphic to $\mathrm{SO}_3(\mathbb{R})$ or $\mathrm{PSL}_2(\mathbb{R})$. Such groups have been investigated in [20] and Theorem 3.2 below has been proved. Therefore Σ is isomorphic to $\mathrm{SO}_3(\mathbb{R})$, $\mathrm{PSL}_2(\mathbb{R})$ or $\mathrm{SL}_2(\mathbb{R})$. □

We give a summary of [20, Theorem 4.6].

Theorem 3.2. *Let $\mathscr{L}$ be a 4-dimensional elation Laguerre plane that admits a non-solvable automorphism group. Then there is a circle K and a closed connected 3-dimensional subgroup Σ in the stabilizer of K such that $\Sigma \cap T$ is discrete and such that Σ acts fixed-point-freely on K.*

(1) *If Σ acts transitively on K, then Σ is isomorphic to* $\mathrm{SO}_3(\mathbb{R})$.

(2) *If Σ is not transitive on K but still acts fixed-point-freely on K, then Σ is isomorphic to either* $\mathrm{PSL}_2(\mathbb{R})$ or $\mathrm{SL}_2(\mathbb{R})$. *In this case, Σ has precisely three orbits on K, one 1-dimensional orbit O_1 homeomorphic to the 1-sphere $\mathbb{S}_1$ and the two connected components of $K \backslash O_1$. The effective operation of Σ on O_1 is transitive and thus is equivalent to the standard operation of* $\mathrm{PSL}_2(\mathbb{R})$ *on $\mathbb{S}_1$. In case $\Sigma \cong \mathrm{PSL}_2(\mathbb{R})$, the elation group splits into the direct sum of two 3-dimensional irreducible Σ-invariant subspaces; in case $\Sigma \cong \mathrm{SL}_2(\mathbb{R})$, the group Σ acts irreducibly on Δ and the centre of Σ acts trivially on K.*

By Theorem 2.5 we obtain semi-classical Laguerre planes in the above situation unless $\Sigma \cong \mathrm{SL}_2(\mathbb{R})$. We investigate this case in the following and show that this leads to a single candidate for a Laguerre plane. In the next section we then verify the axioms of a 4-dimensional Laguerre plane for the incidence structure found in this section.

3.3. General assumptions for the remainder of this section. Let $\mathscr{L}$ be a 4-dimensional elation Laguerre plane with elation group $\Delta \cong \mathbb{R}^6$. We represent $\mathscr{L}$ as in 2.3 by a matrix-valued map D. Let Σ be a closed connected subgroup of the automorphism group Γ of $\mathscr{L}$ such that

– $\Sigma \cong \mathrm{SL}_2(\mathbb{R})$;
– Σ fixes the circle $K_0 = \{(z,0) | z \in \mathbb{S}_2\}$;
– Σ has a 1-dimensional orbit $O_1 \approx \mathbb{S}_1$ in K_0 and acts transitively on both connected components of $K_0 \setminus O_1$;
– Σ acts irreducibly and faithfully on $\mathscr{C} \approx \Delta \cong \mathbb{R}^6$;
– the centre $Z = \{I_2, -I_2\}$ of Σ acts trivially on K_0;
– the 1-dimensional orbit O_1 contains the points $(\infty, 0)$ and $(0,0)$;
– Σ/Z acts on O_1 like the group $\mathrm{PSL}_2(\mathbb{R})$ on $\mathbb{S}_1$ in its standard transitive action;
– $\Sigma^1_{(\infty,0)} \cong \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} | a,b,d \in \mathbb{R}, ad = 1, a > 0 \right\}$;
– $\Sigma^1_{(0,0)} \cong \left\{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} | a,c,d \in \mathbb{R}, ad = 1, a > 0 \right\}$.

Up to equivalence there is a unique real linear irreducible faithful representation ρ of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{R}^6$. For example, one obtains such a representation ρ on the space of homogeneous polynomials of degree 5 in two variables X and Y by defining $\rho\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for $a,b,c,d \in \mathbb{R}$, $ad - bc = 1$, to be the linear map that takes a homogeneous polynomial $f(X, Y)$ to the polynomial $f(aX + bY, cX + dY)$. With respect to the basis

$$X^5, X^4Y, X^3Y^2, X^2Y^3, XY^4, Y^5$$

one obtains for $\rho\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the following 6×6 matrix

$$\begin{pmatrix} a^5 & 5a^4b & 10a^3b^2 & 10a^2b^3 & 5ab^4 & b^5 \\ a^4c & a^3(ad+4bc) & a^2b(4ad+6bc) & ab^2(6ad+4bc) & b^3(4ad+bc) & b^4d \\ a^3c^2 & a^2c(2ad+3bc) & a(a^2d^2+6abcd+3b^2c^2) & b(3a^2d^2+6abcd+b^2c^2) & b^2d(3ad+2bc) & b^3d^2 \\ a^2c^3 & ac^2(3ad+2bc) & c(3a^2d^2+6abcd+b^2c^2) & d(a^2d^2+6abcd+3b^2c^2) & bd^2(2ad+3bc) & b^2d^3 \\ ac^4 & c^3(4ad+bc) & c^2d(6ad+4bc) & cd^2(4ad+6bc) & d^3(ad+4bc) & bd^4 \\ c^5 & 5c^4d & 10c^3d^2 & 10c^2d^3 & 5cd^4 & d^5 \end{pmatrix}$$

We assume that

– the action of Σ on $\mathscr{C}$ is given by ρ, that is, the coordinate vector $c \in \mathbb{R}^6$ of a circle $K_c \in \mathscr{C}$ is taken by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to $c \cdot \rho\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Note that there is no loss of generality in making the above assumptions; we just have to coordinatise $\mathscr{L}$ suitably.

It is readily seen that the subspace spanned by $\{e_{7-i},\dots,e_6\}$, $i=1,\dots,6$, is the only i-dimensional vector subspace of $\mathbb{R}^6$ which is invariant under all linear transformations $\rho\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}$ for $b\in\mathbb{R}$ where $e_1,\dots,e_6\in\mathbb{R}^6$ are the standard basis vectors for $\mathbb{R}^6$, i.e., e_i is the vector whose i-th entry is 1 and all whose other entries are 0. Since the automorphisms associated with $\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}$ for $b\in\mathbb{R}$ fix $(\infty,0)$, they leave invariant the 2-dimensional subspace $\mathscr{C}_{(\infty,0),(\infty,0)}$ of all circles that touch K_0 at $(\infty,0)$ and the 4-dimensional subspace $\mathscr{C}_{(\infty,0)}$ of all circles through $(\infty,0)$; compare 2.3. Hence $\mathscr{C}_{(\infty,0),(\infty,0)}$ is spanned by e_5 and e_6 and $\mathscr{C}_{(\infty,0)}$ is spanned by e_3, e_4, e_5 and e_6. One similarly finds that the 2-dimensional subspace $\mathscr{C}_{(0,0),(0,0)}$ of all circles that touch K_0 at $(0,0)$ is spanned by e_1 and e_2 and the 4-dimensional subspace $\mathscr{C}_{(0,0)}$ of all circles through $(0,0)$ is spanned by e_1,e_2,e_3 and e_4. Therefore $\mathscr{C}_{(\infty,0),(0,0)}=\mathscr{C}_{(\infty,0)}\cap\mathscr{C}_{(0,0)}$ is spanned by e_3 and e_4. This agrees with the standard interpretation of the matrices $A(z)$ and $B(z)$ in the matrix representation of a 4-dimensional elation Laguerre plane in terms of bundles of circles as given in 2.3.

Finite points of $\mathscr{L}$ will be denoted by (z, w) with $z,w\in\mathbb{R}^2$ or by (x, y, u, v) for $x,y,u,v\in\mathbb{R}$, where $z=(x,y)$ and $w=(u,v)$, whichever is more convenient at the time. Likewise, we use $c=(c_1,c_2,c_3,c_4,c_5,c_6)\in\mathbb{R}^6$ and $c=(c^1,c^2,c^3)$ with $c^1=(c_1,c_2), c^2=(c_3,c_4), c^3=(c_5,c_6)\in\mathbb{R}^2$ simultaneously for a coefficient vector c of a circle K_c. Then

$$\begin{aligned}K_c &= \{(z,c\cdot D(z))|z\in\mathbb{S}_2\}\\ &= \{(z,c^1A(z)+c^2B(z)+c^3)|z\in\mathbb{R}^2\}\cup\{(\infty,c^1)\}\\ &= \{(z,c_1f(z)+c_2g(z)+c_3h(z)+c_4z+(c_5,c_6))|z\in\mathbb{R}^2\}\cup\{(\infty,(c_1,c_2))\}\end{aligned}$$

where $f(z)$ and $g(z)$ denote the first and second row of $A(z)$, respectively, and $h(z)$ denotes the first row of $B(z)$, that is, $f(z)$, $g(z)$ and $h(z)$ are maps describing the circles K_{e_1}, K_{e_2} and K_{e_3}, respectively. The entries of $h(z)$ will be denoted by $u_h(z)$ and $v_h(z)$, that is, $h(z)=(u_h(z),v_h(z))$, and likewise for $f(z)$ and $g(z)$. Note that the second row of $B(z)$ equals z.

Under the above general assumptions we can give an explicit formula for the action of $\Lambda=\Sigma^1_{(\infty,0)}$ on finite points. In order to distinguish the standard basis vectors for $\mathbb{R}^6$ from the standard basis vectors for $\mathbb{R}^2$, we denote the latter by $e'_1=(1,0)$ and $e'_2=(0,1)$.

Lemma 3.4. *Let* $L=\begin{pmatrix}a & b\\ 0 & d\end{pmatrix}\in\mathrm{SL}_2(\mathbb{R})$ *and let* $\lambda\in\Lambda$ *be the associated automorphism of* $\mathscr{L}$. (*Note that* $d=a^{-1}$.) *Then* λ *acts on the set of finite points* $\mathbb{R}^4$ *as follows*:

$$\lambda:(z,w)\mapsto(d^3zL-(2bd,b^2d^2),d^4wL).$$

Proof. The point (z, w) lies on the circles $K_{(0,0,w)}$ and $K_{(0,e_2',w-z)}$. Since

$$\rho(L) = \begin{pmatrix} a^5 & 5a^4b & 10a^3b^2 & 10a^2b^3 & 5ab^4 & b^5 \\ 0 & a^3 & 4a^2b & 6ab^2 & 4b^3 & b^4d \\ 0 & 0 & a & 3b & 3b^2d & b^3d^2 \\ 0 & 0 & 0 & d & 2bd^2 & b^2d^3 \\ 0 & 0 & 0 & 0 & d^3 & bd^4 \\ 0 & 0 & 0 & 0 & 0 & d^5 \end{pmatrix},$$

these circles are mapped under λ to $K_{(0,0,d^4wL)}$ and $K_{(0,de_2',d^4(w-z)L+(2bd^2,b^2d^3))}$, respectively. The latter two circles have precisely one finite point (z', w') in common. One finds $w' = d^4wL$ and $dz' + d^4(w-z)L + (2bd^2, b^2d^3) = d^4wL$. From the latter equation one obtains $z' = d^3zL - (2bd, b^2d^2)$. Thus (z, w) is mapped to $(z', w') = (d^3zL - (2bd, b^2d^2), d^4wL)$ under λ. □

As an immediate consequence of the foregoing Lemma we can explicitly describe the orbits of Σ on the distinguished circle K_0.

Corollary 3.5. *The* 1*-dimensional orbit* O_1 *of* Σ *in* K_0 is

$$O_1 = \left\{ \left(x, -\frac{1}{4}x^2, 0, 0 \right) \middle| x \in \mathbb{R} \right\} \cup \{(\infty, 0)\}$$

and the two 2*-dimensional orbits of* Σ *in* K_0 *are*

$$\{(x, y, 0, 0) | x, y \in \mathbb{R}, 4y + x^2 > 0\}$$

and

$$\{(x, y, 0, 0) | x, y \in \mathbb{R}, 4y + x^2 < 0\}.$$

We are now able to obtain equations for the component functions of $A(z)$ and $B(z)$ for $z \in \mathbb{R}^4$ with only a few parameters.

Proposition 3.6. *Let* $\epsilon = \epsilon(x, y) \in \{+1, 0, -1\}$ *be the sign of* $4y + x^2$ *and let*

$$C(x, y) = \begin{pmatrix} |y + \frac{1}{4}x^2| & -\frac{1}{2}x|y + \frac{1}{4}x^2| \\ 0 & |y + \frac{1}{4}x^2|^{3/2} \end{pmatrix}.$$

Then the component functions of $A(z)$ *and* $B(z)$ *satisfy the following identities*:

$$h(x, y) = h(0, \epsilon)C(x, y) + \left(\frac{3}{4}x^2, \frac{3}{2}xy + \frac{1}{8}x^3 \right),$$

$$g(x, y) = \left(g(0, \epsilon) \left| y + \frac{1}{4}x^2 \right|^{1/2} + 2xh(0, \epsilon) \right) C(x, y) + \left(\frac{1}{2}x^3, \frac{3}{2}x^2y + \frac{3}{16}x^4 \right),$$

$$f(x, y) = \left(f(0, \epsilon) \left| y + \frac{1}{4}x^2 \right| + \frac{5}{2}xg(0, \epsilon) \left| y + \frac{1}{4}x^2 \right|^{1/2} + \frac{5}{2}x^2h(0, \epsilon) \right) C(x, y)$$
$$+ \left(\frac{5}{16}x^4, \frac{5}{4}x^3y + \frac{3}{16}x^5 \right).$$

Proof. We first consider the first row $h(z)$ of $B(z)$. The corresponding circle is K_{e_3}. This circle is mapped to $K_{(0,0,a,3b,3b^2d,b^3d^2)}$ under $\rho\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$. Using Lemma 3.4 we thus obtain a functional equation for $h(z) = h(x,y)$ where $z = (x,y)$. The point $(z, h(z)) \in K_{e_3}$ is mapped to $\left(d^3 z\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} - (2bd, b^2d^2), d^4 h(z)\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right)$, which must be on $K_{(0,0,a,3b,3b^2d,b^3d^2)}$. Hence

$$\begin{aligned} d^4 h(z)\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = {} & ah\left(d^3 z\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} - (2bd, b^2d^2)\right) \\ & + 3b\left(d^3 z\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} - (2bd, b^2d^2)\right) \\ & + (3b^2d, b^3d^2) \end{aligned}$$

for all $z \in \mathbb{R}^2$, $a, b, d \in \mathbb{R}, ad = 1, a > 0$. Given $z = (x,y)$ we let $a = d = 1$ and $b = \frac{x}{2}$. Then $d^3 z\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} - (2bd, b^2d^2) = \left(0, y + \frac{1}{4}x^2\right)$ and

$$\begin{aligned} h(x,y) &= \left(h\left(0, y + \frac{1}{4}x^2\right) + 3b\left(0, y + \frac{1}{4}x^2\right) + (3b^2, b^3)\right)\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}^{-1} \\ &= h\left(0, y + \frac{1}{4}x^2\right)\begin{pmatrix} 1 & -\frac{x}{2} \\ 0 & 1 \end{pmatrix} + \left(\frac{3}{4}x^2, \frac{3}{2}xy + \frac{1}{8}x^3\right). \end{aligned}$$

We now let $x = b = 0$. Then

$$d^4 h(0,y)\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = ah(0, d^4 y)$$

and thus

$$h(0, d^4 y) = h(0,y)\begin{pmatrix} d^4 & 0 \\ 0 & d^6 \end{pmatrix}$$

for all $y, d \in \mathbb{R}, d > 0$. Combining these two equations, we obtain the equation for $h(x, y)$ in the proposition.

Likewise, one obtains the equations for $g(z)$ and $f(z)$ by considering the circles K_{e_2} and K_{e_1}, respectively. K_{e_2} is mapped to $K_{(0,a^3,4a^2b,6ab^2,4b^3,b^4d)}$ under $\rho\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$. Therefore

$$\begin{aligned} d^4 g(z)\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = {} & a^3 g\left(d^3 z\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} - (2bd, b^2d^2)\right) \\ & + 4a^2 bh\left(d^3 z\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} - (2bd, b^2d^2)\right) \\ & + 6ab^2\left(d^3 z\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} - (2bd, b^2d^2)\right) \\ & + (4b^3, b^4 d) \end{aligned}$$

for all $z\in\mathbb{R}^2$, $a,b,d\in\mathbb{R}$, $ad=1$, $a>0$. For $a=d=1$ and $b=\frac{x}{2}$ we then find

$$g(x,y)=\left(g\left(0,y+\frac{1}{4}x^2\right)+2xh\left(0,y+\frac{1}{4}x^2\right)\right)\begin{pmatrix}1 & -\frac{x}{2}\\ 0 & 1\end{pmatrix}$$
$$+\left(\frac{1}{2}x^3,\frac{3}{2}x^2y+\frac{3}{16}x^4\right)$$

for all $x,y\in\mathbb{R}$ and $b=x=0$ yields

$$g(0,d^4y)=g(0,y)\begin{pmatrix}d^6 & 0\\ 0 & d^8\end{pmatrix}$$

for all $y,d\in\mathbb{R},d>0$. Taking the two equations together we obtain the stated identity for $g(x,y)$.

Finally, K_{e_1} is mapped to $K_{(a^5,5a^4b,10a^3b^2,10a^2b^3,5ab^4,b^5)}$ under $\rho\begin{pmatrix}a & b\\ 0 & d\end{pmatrix}$. Therefore

$$d^4f(z)\begin{pmatrix}a & b\\ 0 & d\end{pmatrix}=a^5f\left(d^3z\begin{pmatrix}a & b\\ 0 & d\end{pmatrix}-(2bd,b^2d^2)\right)$$
$$+5a^4bg\left(d^3z\begin{pmatrix}a & b\\ 0 & d\end{pmatrix}-(2bd,b^2d^2)\right)$$
$$+10a^3b^2h\left(d^3z\begin{pmatrix}a & b\\ 0 & d\end{pmatrix}-(2bd,b^2d^2)\right)$$
$$+10a^2b^3\left(d^3z\begin{pmatrix}a & b\\ 0 & d\end{pmatrix}-(2bd,b^2d^2)\right)$$
$$+(5ab^4,b^5)$$

for all $z\in\mathbb{R}^2$, $a,b,d\in\mathbb{R}$, $ad=1$, $a>0$. For $a=d=1$ and $b=\frac{x}{2}$ we then find

$$f(x,y)=\left(f\left(0,y+\frac{1}{4}x^2\right)+\frac{5}{2}xg\left(0,y+\frac{1}{4}x^2\right)+\frac{5}{2}x^2h\left(0,y+\frac{1}{4}x^2\right)\right)\begin{pmatrix}1 & -\frac{x}{2}\\ 0 & 1\end{pmatrix}$$
$$+\left(\frac{5}{16}x^4,\frac{5}{4}x^3y+\frac{3}{16}x^5\right)$$

for all $x,y\in\mathbb{R}$ and $b=x=0$ yields

$$f(0,d^4y)=f(0,y)\begin{pmatrix}d^8 & 0\\ 0 & d^{10}\end{pmatrix}$$

for all $y,d\in\mathbb{R},d>0$. Taking the two equations together we obtain the stated identity for $f(x,y)$. □

The proposition shows that the circles of the Laguerre plane $\mathscr{L}$ are uniquely determined by the values of $f(0,\pm 1)$, $g(0,\pm 1)$ and $h(0,\pm 1)$. In the following proposition we obtain some restrictions on the values of $h(0,\pm 1)$ and show that $f(0,\pm 1)$ and $g(0,\pm 1)$ are in fact determined by $h(0,\pm 1)$.

Proposition 3.7. *Let $\delta = \pm 1$ and let $h(0,\delta) = (u_h^\delta, v_h^\delta)$, $g(0,\delta) = (u_g^\delta, v_g^\delta)$ and $f(0,\delta) = (u_f^\delta, v_f^\delta)$. Then*

$$\delta u_h^\delta > 0 \quad \textit{and} \quad u_h^\delta \geqslant 3\delta.$$

Furthermore,

$$u_g^\delta(9\delta - u_h^\delta) = 8u_h^\delta v_h^\delta, \quad v_g^\delta = \delta u_h^\delta + \frac{9v_h^\delta u_g^\delta}{8u_h^\delta},$$
$$u_f^\delta(6\delta - u_h^\delta) = 5u_h^\delta v_g^\delta, \quad v_f^\delta = \delta u_g^\delta + \frac{6v_h^\delta u_f^\delta}{5u_h^\delta}.$$

In particular, $f(0,\delta)$ and $g(0,\delta)$ are completely determined by $h(0,\delta)$ unless $u_h^{+1} = 9$ or $u_h^{+1} = 6$.

Proof. Recall that the matrices $B(z) = \begin{pmatrix} h(z) \\ z \end{pmatrix}$ for $z \in \mathbb{R}^2$ form a spread set. Hence $B(x,y) - B(0,\delta)$ must be regular for all $(x,y) \neq (0,\delta)$. In particular,

$$\det(B(2t,-t^2) - B(0,\delta)) = \det\begin{pmatrix} 3t^2 - u_h^\delta & -v_h^\delta - 2t^3 \\ 2t & -\delta - t^2 \end{pmatrix}$$
$$= t^4 + (u_h^\delta - 3\delta)t^2 + 2v_h^\delta t + \delta u_h^\delta$$

must be non-zero for all $t \in \mathbb{R}$. This implies $\delta u_h^\delta > 0$. Similarly,

$$\det(B(2t,\delta - t^2) - B(0,\delta)) = \det\begin{pmatrix} 3t^2 & -2t^3 - (u_h^\delta - 3\delta)t \\ 2t & -t^2 \end{pmatrix}$$
$$= t^4 + 2(u_h^\delta - 3\delta)t^2$$

must be non-zero for all $t \in \mathbb{R}$. This implies $u_h^\delta \geqslant 3\delta$.

The maps $f(x,y)$, $g(x,y)$ and $h(x,y)$ are differentiable in x and y at all points (x, y) for which $4y + x^2 \neq 0$. Hence, if two circles touch at a point, then their describing functions must have the same derivatives at the point; cf. [17, Remark 5.11.a].

To apply this criterion we first find the partial derivatives of $f(x, y)$, $g(x, y)$ and $h(x, y)$ at $(0,\delta)$. We have $C(0,\delta) = I_2$ and the partial derivatives of $C(x,y)$ at $(0,\delta)$ are $C_y(0,\delta) = \begin{pmatrix} \delta & 0 \\ 0 & \frac{3}{2}\delta \end{pmatrix}$ and $C_x(0,\delta) = \begin{pmatrix} 0 & -\frac{1}{2} \\ 0 & 0 \end{pmatrix}$. One then finds with the product rule

$$\begin{aligned}
h_y(0,\delta) &= h(0,\delta)C_y(0,\delta) &&= \tfrac{1}{2}\delta h(0,\delta)\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} &&= \tfrac{1}{2}\delta(2u_h^\delta, 3v_h^\delta),\\
h_x(0,\delta) &= h(0,\delta)C_x(0,\delta) + (0,\tfrac{3}{2}\delta) &&= \tfrac{1}{2}(0, 3\delta - u_h^\delta),\\
g_y(0,\delta) &= g(0,\delta)C_y(0,\delta) + \tfrac{1}{2}\delta g(0,\delta) &&= \tfrac{1}{2}\delta g(0,\delta)\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} &&= \tfrac{1}{2}\delta(3u_g^\delta, 4v_g^\delta),\\
g_x(0,\delta) &= g(0,\delta)C_x(0,\delta) + 2h(0,\delta) &&= (2u_h^\delta, 2v_h^\delta - \tfrac{1}{2}u_g^\delta),\\
f_y(0,\delta) &= f(0,\delta)C_y(0,\delta) + \delta f(0,\delta) &&= \tfrac{1}{2}\delta f(0,\delta)\begin{pmatrix} 4 & 0 \\ 0 & 5 \end{pmatrix} &&= \tfrac{1}{2}\delta(4u_f^\delta, 5v_f^\delta),\\
f_x(0,\delta) &= f(0,\delta)C_x(0,\delta) + \tfrac{5}{2}g(0,\delta) &&= (\tfrac{5}{2}u_g^\delta, \tfrac{5}{2}v_g^\delta - \tfrac{1}{2}u_f^\delta).
\end{aligned}$$

We now consider the circle K_{e_2} described by the map $g(z)$. Let $K_{(0,r,s)}$, $r,s \in \mathbb{R}^2$, be the circle through $(\infty, 0)$ that touches K_{e_2} at $((0,\delta), g(0,\delta))$. Then $g(z)$ and $rB(z) + s$ have the same derivative at $(0,\delta)$. Therefore $rB_y(0,\delta) = g_y(0,\delta)$ and $rB_x(0,\delta) = g_x(0,\delta)$. From the above we find that

$$B_y(0,\delta) = \begin{pmatrix} \delta u_h^\delta & \frac{3}{2}\delta v_h^\delta \\ 0 & 1 \end{pmatrix}.$$

Since $\delta u_h^\delta > 0$, we see that $B_y(0,\delta)$ is regular. Hence $r = g_y(0,\delta)B_y(0,\delta)^{-1}$ and therefore

$$g_y(0,\delta)B_y(0,\delta)^{-1}B_x(0,\delta) = g_x(0,\delta).$$

Now

$$B_y(0,\delta)^{-1}B_x(0,\delta) = \frac{1}{2u_h^\delta}\begin{pmatrix} -3v_h^\delta & 3-\delta u_h^\delta \\ 2u_h^\delta & 0 \end{pmatrix}.$$

Thus

$$\begin{aligned} 8(u_h^\delta)^2 &= \delta(8u_h^\delta v_g^\delta - 9v_h^\delta u_g^\delta) \\ 2(4v_h^\delta - u_g^\delta)u_h^\delta &= 3\delta(3-\delta u_h^\delta)u_g^\delta \end{aligned}$$

The second equation yields

$$u_g^\delta(9\delta - u_h^\delta) = 8u_h^\delta v_h^\delta$$

and the first equation yields the expression for v_g^δ.

One likewise obtains that

$$f_y(0,\delta)B_y(0,\delta)^{-1}B_x(0,\delta) = f_x(0,\delta)$$

and one finds

$$\begin{aligned} 5u_h^\delta v_g^\delta &= \delta(5u_h^\delta v_f^\delta - 6v_h^\delta)u_f^\delta, \\ (5v_g^\delta - u_f^\delta)u_h^\delta &= 2\delta(3-\delta u_h^\delta)u_f^\delta. \end{aligned}$$

From these two equations the respective identities for u_f^δ and v_f^δ follow.

$u_h^\delta = 9\delta$ or $u_h^\delta = 6\delta$ can only occur if $\delta = +1$ since $u_h^\delta \geqslant 3\delta$. In all other cases we can solve the equations for u_g^δ and u_f^δ. Substituting the values for u_g^δ and v_g^δ into the formulae for u_f^δ and v_f^δ, we see that u_g^δ, v_g^δ, u_f^δ, and v_f^δ are completely determined by u_h^δ and v_h^δ. □

The derived affine plane at $(\infty, 0)$ is a dual translation plane. Dualising this plane we obtain a (projective) translation plane $\mathscr{P}$. Since the group Λ induces a group of collineations of $\mathscr{P}$, the collineation group $\mathscr{P}$ is at least 7-dimensional. $\mathscr{P}$ cannot be Desarguesian as this implies that $\mathscr{L}$ is classical. So the collineation group of $\mathscr{P}$ is either 8- or 7-dimensional. All 4-dimensional translation planes admitting collineation groups of dimension 7 or 8 have been classified by Betten, see [12, §73] for an overview. If the collineation group of $\mathscr{P}$ is 7-dimensional, then $\mathscr{P}$ is isomorphic to one of the planes in [2, Satz 5].

So far we have not used the full group Σ. It is well known that $\Sigma \cong \mathrm{SL}_2(\mathbb{R})$ is generated by $\Lambda \cong \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} | a,b,d \in \mathbb{R}, ad = 1, a > 0 \right\}$ and the automorphism σ associated with the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. In order to know how Σ acts on $\mathscr{L}$ it suffices to know the action of σ.

Lemma 3.8. *Let* $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ *and let* $\sigma \in \Sigma$ *be the associated automorphism of* $\mathscr{L}$. *Then* σ *acts on the finite points* (z, w) *with* $z \neq 0$ *as follows*:

$$\sigma : (z,w) \mapsto (h(z)A(z)^{-1}S, wA(z)^{-1}S).$$

Proof. The point $(z,w), z \neq 0$, is on the circle $K_{(r,0,0)}$ as well as on the circle $K_{(s,e_1,0)}$ where $r = wA(z)^{-1}$ and $s = (w - h(z))A(z)^{-1}$. Both circles also have the point $(0,0)$ in common. Since $\sigma\Lambda\sigma^{-1} = \Sigma^1_{(0,0)}$, the point $(0,0)$ must be mapped to $(\infty, 0)$ under σ. Hence we find the image of (z, w) as the finite point of intersection of the images of $K_{(r,0,0)}$ and $K_{(s,e_1',0)}$. Since $\rho(S) = \begin{pmatrix} 0 & 0 & S \\ 0 & S & 0 \\ S & 0 & 0 \end{pmatrix}$, one finds that $K_{(r,0,0)}$ is mapped to $K_{(0,0,rS)}$ and $K_{(s,e_1',0)}$ is mapped to $K_{(0,e_2',sS)}$. These two circles intersect in $(\infty,0)$ and $(h(z)A(z)^{-1}S, wA(z)^{-1}S)$. □

We can now put the pieces together and show that there is at most one candidate for the kind of Laguerre plane we are investigating in this section.

Theorem 3.9. *There is at most one elation Laguerre plane admitting* $\mathrm{SL}_2(\mathbb{R})$ *as a group of automorphisms. In this plane one has*

$$f(0,1) = (125,0), \quad g(0,1) = (0,5), \quad h(0,1) = (5,0),$$
$$f(0,-1) = (1,0), \quad g(0,-1) = (0,1), \quad h(0,-1) = (-1,0),$$

that is,

$$h(x,y) = \left(5y + 2x^2, -x\left(y + \frac{1}{2}x^2\right)\right),$$

$$g(x,y) = \left(x(10y + 3x^2), \left(5y + \frac{3}{2}x^2\right)\left(y - \frac{1}{2}x^2\right)\right),$$

$$f(x,y) = \left(5\left(5y + \frac{3}{2}x^2\right)^2, -2x\left(5y + \frac{3}{2}x^2\right)^2\right),$$

for $4y + x^2 \geqslant 0$, *and*

$$h(x,y) = (y + x^2, xy),$$
$$g(x,y) = (x(2y + x^2), y(y + x^2)),$$
$$f(x,y) = (y^2 + 3x^2y + x^4, xy(2y + x^2)),$$

for $4y + x^2 \leqslant 0$.

Proof. Since $\rho(S) = \begin{pmatrix} 0 & 0 & S \\ 0 & S & 0 \\ S & 0 & 0 \end{pmatrix}$, the circle K_{e_4} is mapped under σ to K_{-e_3}. Hence $(h(z)A(z)^{-1}S, zA(z)^{-1}S) = \sigma(z,z) \in K_{-e_3}$ for all $z \in \mathbb{R}^2$, $z \neq 0$ and therefore

$$h(h(z)A(z)^{-1}S) = -zA(z)^{-1}S$$

for all $z \in \mathbb{R}^2$, $z \neq 0$. We use this identity for points $z = (2t, \delta - t^2)$ for $t \in \mathbb{R}$. The first components of both sides when multiplied by a suitable function give us polynomials in t. Comparing corresponding coefficients and using Proposition 3.7 we shall be able to determine $h(0,\delta) = (u_h^\delta, v_h^\delta)$ and thus $g(0,\delta) = (u_g^\delta, v_g^\delta)$ and $f(0,\delta) = (u_f^\delta, v_f^\delta)$.

One finds from Proposition 3.6 with $C(2t, \delta - t^2) = \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}$ that

$$\begin{aligned}
h(2t, \delta - t^2) &= (u_h^\delta + 3t^2, v_h^\delta + (3\delta - u_h^\delta)t - 2t^3), \\
g(2t, \delta - t^2) &= (u_g^\delta + 4u_h^\delta t + 4t^3, v_g^\delta + (4v_h^\delta - u_g^\delta)t + 2(3\delta - 2u_h^\delta)t^2 - 3t^4), \\
f(2t, \delta - t^2) &= (u_f^\delta + 5u_g^\delta t + 10u_h^\delta t^2 + 5t^4, \\
&\qquad v_f^\delta + (5v_g^\delta - u_f^\delta)t + 5(2v_h^\delta - u_g^\delta)t^2 + 10(\delta - u_h^\delta)t^3 - 4t^5).
\end{aligned}$$

One further computes

$$h(2t, \delta - t^2)A(2t, \delta - t^2)^{-1}S = \frac{1}{d}(\zeta, \eta)$$

where

$$\begin{aligned}
d &= \det A(2t, \delta - t^2) \\
&= u_f^\delta v_g^\delta - u_g^\delta v_f^\delta + 4(u_f^\delta v_h^\delta - u_h^\delta v_f^\delta)t + 2(3\delta u_f^\delta + 5(u_g^\delta v_h^\delta - u_h^\delta v_g^\delta))t^2 \\
&\quad + 4(5\delta u_g^\delta - v_f^\delta)t^3 + (u_f^\delta - 15v_g^\delta + 20\delta u_h^\delta)t^4 + 4(u_g^\delta - 5v_h^\delta)t^5 \\
&\quad + 2(3u_h^\delta - 5\delta)t^6 + t^8, \\
\zeta &= u_h^\delta v_f^\delta - u_f^\delta v_h^\delta + (5(u_h^\delta v_g^\delta - u_g^\delta v_h^\delta) - 3\delta u_f^\delta)t + 3(v_f^\delta - 5\delta u_g^\delta)t^2 \\
&\quad + (15v_g^\delta - u_f^\delta - 20\delta u_h^\delta)t^3 + 5(5v_h^\delta - u_g^\delta)t^4 + 3(5\delta - 3u_h^\delta)t^5 - 2t^7, \\
\eta &= u_h^\delta v_g^\delta - u_g^\delta v_h^\delta - 3\delta u_g^\delta t + 3(v_g^\delta - 2\delta u_h^\delta)t^2 + (8v_h^\delta - u_g^\delta)t^3 \\
&\quad + 3(2\delta - u_h^\delta)t^4 - t^6.
\end{aligned}$$

Now the first component of $h(\frac{\zeta}{d}, \frac{\eta}{d})$ is $\frac{1}{4d^2}(3\zeta^2 + u_h^\delta|4d\eta + \zeta^2|)$ which must be equal to the first component of $-(2t, \delta - t^2)A(2t, \delta - t^2)^{-1}S$, that is,

$$\begin{aligned}
\frac{1}{4d^2}(3\zeta^2 + |4d\eta + \zeta^2|) = \frac{1}{d}(&\delta u_f^\delta + (5\delta u_g^\delta - 2v_f^\delta)t + (u_f^\delta - 10v_g^\delta + 10\delta u_h^\delta)t^2 \\
&+ 5(u_g^\delta - 4v_h^\delta)t^3 + 5(2u_h^\delta - 3\delta)t^4 + 3t^6)
\end{aligned}$$

for all $t \in \mathbb{R}$. Note that $4d\eta + \zeta^2$ must have the same sign as δ so that $|4d\eta + \zeta^2| = \delta(4d\eta + \zeta^2)$. Multiplying both sides by $4d^2$ we obtain a polynomial equation in t. Expanding we obtain that

$$\begin{aligned}F_\delta(t) &= 4((5+3\delta u_h^\delta)v_h^\delta - 2u_g^\delta)t^{11} + \cdots \\ &\quad + 2[3(5-\delta u_h^\delta)(u_h^\delta v_f^\delta - u_f^\delta v_h^\delta)(u_h^\delta v_g^\delta - u_g^\delta v_h^\delta) \\ &\qquad - u_f^\delta(3u_h^\delta+\delta)(u_h^\delta v_f^\delta - u_f^\delta v_h^\delta) \\ &\qquad - 2(u_f^\delta v_g^\delta - u_g^\delta v_f^\delta)(3u_h^\delta u_g^\delta + 5\delta u_g^\delta - 2v_f^\delta)]t \\ &\quad + (u_h^\delta v_f^\delta - u_f^\delta v_h^\delta)^2(3+\delta u_h^\delta) - 4\delta(u_f^\delta - u_h^\delta(u_h^\delta v_g^\delta - u_g^\delta v_h^\delta))(u_f^\delta v_g^\delta - u_g^\delta v_f^\delta) \\ &= 0\end{aligned}$$

for all $t \in \mathbb{R}$. Hence each coefficient of $F_\delta(t)$ must be 0. Equating the coefficient of t^{11} to 0 we find $u_g^\delta = \frac{1}{2}(5+3\delta u_h^\delta)v_h^\delta$. From Proposition 3.7 we obtain

$$(9\delta - u_h^\delta)(5+3\delta u_h^\delta)v_h^\delta = 16u_h^\delta v_h^\delta.$$

Therefore

$$v_h^\delta = 0 \quad \text{or} \quad (u_h^\delta - 5\delta)(u_h^\delta + 3\delta) = 0.$$

Since $\delta u_h^\delta > 0$, we have $u_h^\delta + 3\delta \neq 0$. Furthermore, because $u_h^\delta \leqslant 3\delta$, the case $u_h^\delta = 5\delta$ can only occur when $\delta = +1$. Hence

$$v_h^\delta = 0 \quad \text{or} \quad u_h^{+1} = 5.$$

We first assume that $v_h^\delta = 0$. Then $u_g^\delta = 0$ and $v_f^\delta = 0$; moreover, $v_g^\delta = \delta u_h^\delta$ and $u_f^\delta = \frac{5(u_h^\delta)^2}{6-\delta u_h^\delta}$ from Proposition 3.7. Substituting these expressions into the constant term of $F_\delta(t)$ and equating to 0 we obtain

$$0 = (u_h^\delta)^2 - 6\delta u_h^\delta + 5 = (u_h^\delta - 3\delta + 2)(u_h^\delta - 3\delta - 2).$$

Since $u_h^\delta \geqslant 3\delta$ by Proposition 3.7, we find that

$$\begin{aligned} u_h^\delta &= 3\delta + 2, & v_h^\delta &= 0, \\ u_g^\delta &= 0, & v_g^\delta &= 3 + 2\delta, \\ u_f^\delta &= 63 + 62\delta, & v_f^\delta &= 0. \end{aligned}$$

This gives us the stated values for $f(0, \pm 1)$, $g(0, \pm 1)$ and $h(0, \pm 1)$.

We finally assume that $u_h^{+1} = 5$. Then

$$\begin{aligned} u_g^{+1} &= 10v_h^{+1}, & v_g^{+1} &= \tfrac{1}{4}(20 + 9(v_h^{+1})^2), \\ u_f^{+1} &= \tfrac{25}{4}(20 + 9(v_h^{+1})^2), & v_f^{+1} &= \tfrac{1}{2}v_h^{+1}(80 + 27(v_h^{+1})^2) \end{aligned}$$

from Proposition 3.7. Substituting these expressions into the coefficient of t^1 of $F_\delta(t)$ and equating to 0 we obtain

$$v_h^{+1}(27(v_h^{+1})^2 + 80)(9(v_h^{+1})^4 - 40(v_h^{+1})^2 + 2000) = 0.$$

Hence $v_h^{+1} = 0$ and we find the same values for $f(0,1)$, $g(0,1)$ and $h(0,1)$ as before. □

Remark 3.10. The circle describing maps h, g and f can be given in closed form as follows:

$$h(x,y) = \left(3y + \frac{3}{2}x^2, -\frac{1}{4}x^3\right) + \left|y + \frac{1}{4}x^2\right|(2,-x),$$

$$g(x,y) = \left(2x(3y+x^2), 3y^2 - \frac{3}{8}x^4\right) + \left|y + \frac{1}{4}x^2\right|\left(4x, 2y - \frac{3}{2}x^2\right),$$

$$f(x,y) = \left(63y^2 + 39x^2y + \frac{49}{8}x^4, -x\left(24y^2 + \frac{29}{2}x^2y + \frac{9}{4}x^4\right)\right)$$
$$+ \left|y + \frac{1}{4}x^2\right|\left(62y + \frac{41}{2}x^2, -x(26y + 9x^2)\right).$$

It is interesting to note that the terms not involving $|y + \frac{1}{4}x^2|$ in the above formulae describe an incidence structure that almost is an elation Laguerre plane. More precisely, let

$$\tilde{h}(x,y) = \left(3y + \frac{3}{2}x^2, -\frac{1}{4}x^3\right),$$

$$\tilde{g}(x,y) = \left(2x(3y+x^2), 3y^2 - \frac{3}{8}x^4\right),$$

$$\tilde{f}(x,y) = \left(63y^2 + 39x^2y + \frac{49}{8}x^4, -x\left(24y^2 + \frac{29}{2}x^2y + \frac{9}{4}x^4\right)\right).$$

Then

$$K_c = \{(z, c_1\tilde{f}(z) + c_2\tilde{g}(z) + c_3\tilde{h}(z) + c_4 z + (c_5, c_6)) | z \in \mathbb{R}^2\} \cup \{(\infty, (c_1, c_2))\}$$

defines a set of circles for $c = (c_1, c_2, c_3, c_4, c_5, c_6) \in \mathbb{R}^6$. The derived incidence structure at $(\infty, 0)$ is a dual translation plane; in fact, this affine plane $\mathscr{A}$ is isomorphic to the dual of Betten's translation plane from [1, Satz 2.b]. The circle K_{e_2} induces an oval in $\mathscr{A}$. However, K_{e_1} does not induce a topological oval in $\mathscr{A}$, since, for example, the equation $\tilde{f}(x,y) = (0,1)$ has no real solution whereas by [3, Satz 3.1] a topological oval in a 4-dimensional projective plane possesses no exterior lines. In fact, the substructure obtained for all circles K_c with $c_1 = 0$ is isomorphic to the corresponding substructure obtained from the single 4-dimensional elation Laguerre plane found in [15, §8].

4. The Axioms of a Laguerre Plane

In this section $\mathscr{L} = (P, \mathscr{C}, \|)$ always denotes the incidence structure we obtained in Theorem 3.9, that is,

$$P = (\mathbb{R}^2 \cup \{\infty\}) \times \mathbb{R}^2$$

and

$$\mathscr{C} = \{\{(z, c^1 A(z) + c^2 B(z) + c^3) | z \in \mathbb{R}^2\} \cup \{(\infty, c^1)\} | c^1, c^2, c^3 \in \mathbb{R}^2\}$$

where

$$A(x,y) = \begin{pmatrix} y^2 + 3x^2y + x^4 & xy(2y + x^2) \\ x(2y + x^2) & y(y + x^2) \end{pmatrix},$$
$$B(x,y) = \begin{pmatrix} y + x^2 & xy \\ x & y \end{pmatrix},$$

for $4y + x^2 \leqslant 0$, and

$$A(x,y) = \begin{pmatrix} 5\big(5y + \frac{3}{2}x^2\big)^2 & -2x\big(5y + \frac{3}{2}x^2\big)^2 \\ x(10y + 3x^2) & \big(5y + \frac{3}{2}x^2\big)\big(y - \frac{1}{2}x^2\big) \end{pmatrix},$$
$$B(x,y) = \begin{pmatrix} 5y + 2x^2 & -x\big(y + \frac{1}{2}x^2\big) \\ x & y \end{pmatrix},$$

for $4y + x^2 \geqslant 0$. Later on it will prove convenient to rewrite the matrices $A(x, y)$ and $B(x, y)$ as follows

$$A(x,y) = \begin{pmatrix} 5q^2 & -2xq^2 \\ 2xq & \frac{1}{5}q(q - 4x^2) \end{pmatrix},$$
$$B(x,y) = \begin{pmatrix} q + \frac{1}{2}x^2 & -\frac{1}{5}x(q + x^2) \\ x & \frac{1}{5}q - \frac{3}{10}x^2 \end{pmatrix},$$

where $q = 5y + \frac{3}{2}x^2$ and $4y + x^2 \geqslant 0$.

We verify the axioms of a topological Laguerre plane, that is, we show

– that three mutually non-parallel points can uniquely be joined by a circle,
– that a circle K and a pair (p, q) of non-parallel points with $p \in K$ determine a unique circle that touches K at p and passes through q, and
– that the geometric operations are continuous.

We begin with a few useful observations, one of which shows that circles look like 'parabolae' on the subset $\{(x, y, u, v) \in \mathbb{R}^4 | 4y + x^2 < 0\}$.

Lemma 4.1. *The matrix-valued functions $A(z)$ and $B(z)$ are continuous on $\mathbb{R}^2$. Let $z = (x, y) \in \mathbb{R}^2$. Then*

$$\det A(z) = \begin{cases} y^4, & \text{for } 4y + x^2 \leqslant 0, \\ \big(5y + \frac{3}{2}x^2\big)^4, & \text{for } 4y + x^2 \geqslant 0. \end{cases}$$

In particular, $A(z)$ is invertible for all $z \in \mathbb{R}^2$, $z \neq 0$.

Furthermore, $A(z) = B(z)^2$ for $4y + x^2 \leqslant 0$.

We now come to two homogeneity properties of $\mathscr{L}$ which allow us later in the verification of the axioms to specialise some of the points involved. A straightforward computation shows that the formulae in Lemmata 3.4 and 3.8 indeed define automorphisms of $\mathscr{L}$ where σ from Lemma 3.8 is extended by $\sigma(0, w) = (\infty, wS)$ and $\sigma(\infty, w) = (0, wS)$. Hence we obtain the following.

Proposition 4.2. $\Sigma \cong \mathrm{SL}_2(\mathbb{R})$ *acts as a group of automorphisms of* $\mathscr{L}$ *and has the three orbits*

$$\{(x,y,0,0) \in \mathbb{R}^4 | 4y + x^2 > 0\},$$
$$\{(x,y,0,0) \in \mathbb{R}^4 | 4y + x^2 < 0\} \text{ and}$$
$$\{(x,y,0,0) \in \mathbb{R}^4 | 4y + x^2 = 0\} \cup \{(\infty,0,0)\}$$

in K_0. *Correspondingly,* $\Sigma \ltimes \Delta$ *has three orbits in the point space, two* 4*-dimensional orbits*

$$O_4^+ = \{(x,y,u,v) \in \mathbb{R}^4 | 4y + x^2 > 0\},$$
$$O_4^- = \{(x,y,u,v) \in \mathbb{R}^4 | 4y + x^2 < 0\}$$

and one 3*-dimensional orbit*

$$O_3 = \{(x,y,u,v) \in \mathbb{R}^4 | 4y + x^2 = 0\} \cup \{\infty\} \times \mathbb{R}^2.$$

Furthermore, the action of Σ *on* $\mathscr{C}$ *is given by* ρ.

By Proposition 4.2 every point in O_4^- can be mapped to the point $(0,-1,0,0)$ by some automorphism. The stabilizer of this point allows us some further reduction.

Lemma 4.3. *Let* $\Phi \cong \mathrm{SO}_2(\mathbb{R})$ *be the canonical subgroup corresponding to the rotation group* $\left\{ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \middle| t \in \mathbb{R} \right\}$. *Then* Φ *has the fixed points* $(0,-1,0,0)$ *and* $\left(0,\frac{1}{5},0,0\right)$ *in* K_0. *Furthermore, every point of* K_0 *can be mapped under* Φ *to a point* $(0,r,0,0)$ *for some* $r \in \mathbb{R}$.

Proof. The automorphism σ belonging to $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is an element of Φ. Using the formula from 3.8 we explicitly find for the action of σ on K_0

$$\sigma : (x,y,0,0) \mapsto \begin{cases} \left(\frac{x}{y}, \frac{1}{y}, 0, 0\right), & \text{for } 4y + x^2 \leqslant 0, (x,y) \neq (0,0), \\ \left(-\frac{2x}{10y+3x^2}, \frac{4y}{(10y+3x^2)^2}, 0, 0\right), & \text{for } 4y + x^2 \geqslant 0, (x,y) \neq (0,0), \\ (\infty,0,0), & \text{for } x = y = 0, \\ (0,0,0,0), & \text{for } (x,y) = \infty. \end{cases}$$

It now readily follows that $(0,-1,0,0)$ and $\left(0,\frac{1}{5},0,0\right)$ are the only fixed points of σ on K_0. Since Φ is abelian and connected, both points must be fixed by every element of Φ. By [11, 6.7.1] the group $\mathrm{SO}_2(\mathbb{R})$ can act on $K_0 \approx \mathbb{S}_2$ only trivially or with two fixed points. Furthermore, in the latter case, every 1-dimensional orbit separates the two fixed points. Therefore $(0,-1,0,0)$ and $\left(0,\frac{1}{5},0,0\right)$ are the only fixed points of Φ. Since $\{(0,r,0,0) | r \in \mathbb{R}, -1 \leqslant r \leqslant 1\}$ is a segment connecting both points, we obtain that every orbit of Φ in K_0 must intersect this segment, that is, every orbit contains a point $(0,r,0,0)$ for some $r \in \mathbb{R}$. □

Proposition 4.4. *The derived incidence structure* $\mathscr{A}_p$ *of* $\mathscr{L}$ *at each point* p *in the* 3*-dimensional orbit* $O_3 = \{(x,y,u,v) \in \mathbb{R}^4 | 4y + x^2 = 0\} \cup \{\infty\} \times \mathbb{R}^2$ *is a* *topological affine plane. In particular, $B(z_1) - B(z_2)$ is invertible for all $z_1, z_2 \in \mathbb{R}^2$, $z_1 \neq z_2$.*

Proof. By Proposition 4.2 it suffices to verify the derived incidence structure at $p = (\infty, 0)$ is a topological affine plane. In this case, $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} B(z)^t \begin{pmatrix} -\frac{2}{3} & 0 \\ 0 & 1 \end{pmatrix} \middle| z \in \mathbb{R}^2 \right\}$, where $B(z)^t$ denotes the transpose of $B(z)$, is one of the spread sets described in [2, Satz 5] – in the notation of [2] it is the one with parameters $w = p = 0$, $z = -\frac{3}{5}$, $q = 3$. Hence $\{B(z) | z \in \mathbb{R}^2\}$ is a spread set and $\mathscr{A}_p$ is an affine plane, in fact, a dual translation plane, cf. 2.3. Furthermore, with respect to the induced topologies, $\mathscr{A}_p$ is a topological 4-dimensional affine plane by [2, Satz 5].

Finally, $B(z_1) - B(z_2)$ is invertible for all $z_1, z_2 \in \mathbb{R}^2$, $z_1 \neq z_2$, by the definition of a spread set. □

Proposition 4.5. *Three mutually non-parallel points of $\mathscr{L}$ can be uniquely joined by a circle.*

Proof. Let (z_j, w_j), $j = 1, 2, 3$, three mutually non-parallel points. In view of Proposition 4.4 we can assume that none of the points is in the 3-dimensional orbit O_3; in particular, $z_j \neq \infty$. The coefficient vector $c = (c^1, c^2, c^3) \in \mathbb{R}^6$ of a circle K_c that passes through the three points satisfies the system of linear equations

$$c \cdot \begin{pmatrix} A(z_1) & A(z_2) & A(z_3) \\ B(z_1) & B(z_2) & B(z_3) \\ I_2 & I_2 & I_2 \end{pmatrix} = (w_1, w_2, w_3).$$

We show that the coefficient matrix

$$D(z_1, z_2, z_3) = \begin{pmatrix} A(z_1) & A(z_2) & A(z_3) \\ B(z_1) & B(z_2) & B(z_3) \\ I_2 & I_2 & I_2 \end{pmatrix}$$

is invertible. This then implies the existence and uniqueness of the circle joining the three points. We subtract the third column from the first and second columns and then subtract $(B(z_2) - B(z_3))^{-1}(B(z_1) - B(z_3))$ times the second column (where this column is multiplied by the matrix on the right) from the first column. After these elementary column operations we obtain the matrix

$$\begin{pmatrix} C(z_1, z_2, z_3) & A(z_2) - A(z_3) & A(z_3) \\ 0 & B(z_2) - B(z_3) & B(z_3) \\ 0 & 0 & I_2 \end{pmatrix}$$

where

$$C(z_1, z_2, z_3) = A(z_1) - A(z_3) - (A(z_2) - A(z_3))(B(z_2) - B(z_3))^{-1}(B(z_1) - B(z_3)).$$

Hence

$$\det D(z_1, z_2, z_3) = \det C(z_1, z_2, z_3) \det(B(z_2) - B(z_3))$$