2-Generator Conditions in Linear Groups

By

B. A. F. Wehrfritz

In [4] V. P. Platonov states "a linear group over a field of characteristic zero in which every 2-generator subgroup is soluble-by-finite is itself soluble-by-finite". His proof contains a flaw since the 2-variable law he produces contains more than two variables. The proof can be repaired since a law of the required type must exist; the class of all groups each of whose 2-generator subgroups lies in the product of the variety of soluble groups of derived length at most r and the variety generated by the symmetric group on s symbols is a variety ([2] 16.21) which trivially is not the variety of all groups.

Platonov's proof relies on a relatively deep result of Chevalley ([1] Prop. 23.2). We give another proof (in fact of a little more) but use even deeper results of J. G. Thompson, namely the classification of the (finite) minimal simple groups. It follows from this classification that every such group is 2-generator. We then give two proofs of a nilpotent analogue of Platonov's result which turns out to be a little more subtle.

Lemma 1. *Let R be a finitely generated integral domain and G a subgroup of $GL(n, R)$ such that every 2-generator subgroup of G is soluble-by-finite. Then G is soluble-by-finite.*

Proof*). G contains a normal subgroup H of finite index such that H is residually nilpotent ([6] 4.7). It follows from [5] 6.25 that a residually nilpotent group is soluble if it is soluble-by-finite. Hence every 2-generator subgroup of H is soluble and consequently every finite image of H is soluble by Thompson's results. By Mal'cev's Theorem, [6] 4.2, H is residually linear of degree n and thus is residually soluble of bounded degree, [6] 3.7. Therefore H is soluble.

Throughout F denotes a (commutative) field.

Lemma 2. *Let G be a subgroup of $GL(n, F)$ and $\mathfrak{V}$ a variety of groups such that every finitely generated subgroup of G is a finite extension of a $\mathfrak{V}$-group. Then G is an extension of a $\mathfrak{V}$-group by a periodic linear group over F.*

Proof. Let V be a $\mathfrak{V}$-subgroup of $GL(n, F)$ then $\mathscr{A}_F(V)$, the closure of V in $GL(n,F)$, is also a $\mathfrak{V}$-group. Hence if H is any finitely generated subgroup of G then $\mathscr{A}_F(H^0) = \mathscr{A}_F(H)^0$, the connected component of $\mathscr{A}_F(H)$ containing 1, is a $\mathfrak{V}$-group. Pick a

*) It is also possible to adapt Platonov's method to prove this lemma.

finitely generated subgroup H of G such that $\mathscr{A}_F(H)^0$ has maximal dimension. For each g in G, $\mathscr{A}_F(H)^0 = \mathscr{A}_F(\langle g, H\rangle)^0$ and thus $S = G \cap \mathscr{A}_F(H)^0$ is a closed normal $\mathfrak{B}$-subgroup of G. Also since $\mathscr{A}_F(H)^0$ has finite index in $\mathscr{A}_F(\langle g, H\rangle)$, some positive power of g lies in S and G/S is periodic. Finally G/S is isomorphic to a linear group over F ([6] 6.4).

Theorem 1. *If G is a subgroup of $GL(n, F)$ such that every 2-generator subgroup of G is soluble-by-finite, then G is soluble-by-periodic and if* char $F = 0$ *then G is even soluble-by-finite.*

Proof. By Lemma 1 every finitely generated subgroup of G is soluble-by-finite and hence by Lemma 2, G is soluble-by-periodic. If char $F = 0$ then every periodic linear group over F is abelian-by-finite (Schur's Theorem [6] 9.4) and the theorem follows.

We clearly need to restrict the characteristic for the last part of the theorem since there exist infinite simple periodic linear groups. The following corollary is an immediate consequence of the theorem and [6] 10.3.

Corollary. *Let R be a Noetherian (commutative) ring, M a finitely generated R-module and G a group of R-automorphisms of M. If every 2-generator subgroup of G is soluble-by-finite then G is soluble-by-periodic.*

Theorem 2. *Let G be a subgroup of $GL(n, F)$ and suppose that every 2-generator subgroup of G is nilpotent-by-finite. Then G is nilpotent-by-periodic.*

First proof. We may clearly assume that F is algebraically closed. By Theorem 1, G is soluble-by-periodic, so we may suppose that G is triangularizable ([6] 3.6). Then $\bar{G} = \mathscr{A}_F(G)$ is also triangularizable. Denote by V the maximal unipotent subgroup of $\bar{G}$ and put $U = G \cap V$.

Let $H = \langle g_1, \ldots, g_r\rangle$ be a finitely generated subgroup of G. If $u \in U$ then $\langle u, g_i\rangle$ is nilpotent-by-finite and so $\mathscr{A}_F(\langle u, g_i\rangle)^0$ is nilpotent. Therefore

$$(V \cap \mathscr{A}_F(\langle u, g_i\rangle))\, \mathscr{A}_F(\langle u, g_i\rangle)^0$$

is also nilpotent. By [6] 7.3, $(g_i)_d \in \mathscr{A}_F(\langle u, g_i\rangle)$ and in a nilpotent linear group unipotent elements commute with d-elements ([6] 7.11). Hence for each u in U there exists an integer $k(u)$ such that for $i = 1, 2, \ldots, r$ we have $[u, (g_i)_d^{k(u)}] = 1$.

Let $K = \langle V, (g_i)_d : i = 1, 2, \ldots, r\rangle$ and for any integer k let

$$K_k = \langle V, ((g_i)_d)^k : i = 1, 2, \ldots, r\rangle .$$

Since $\bar{G}/V$ is abelian $(K : K_k)$ is finite for every k. If $u \in U$ then $V C_{\bar{G}}(u)$ is closed in $\bar{G}$, and clearly $K_{k(u)} \subseteqq V C_{\bar{G}}(u)$. Therefore $K^0 \subseteqq V C_{\bar{G}}(u)$ for each u in U.

Let $l = (K : K^0)$. Since $(g_i)_u \in V$ for each i we have $H \subseteqq K$ and $H^l \subseteqq K^0$. Hence $[U, H^l] \subseteqq [U, V] \cap G \subseteqq U$. Now V is unipotent and so is nilpotent of class at most $m = (n-1)$. Therefore $[U,_m H^l] \subseteqq [U,_m V] = \{1\}$ and H^l is nilpotent of class at most $m + 1$ (in fact at most m by [6] 7.11, unless $n = 1$). But H is soluble and finitely generated, so H/H^l is finite. Thus every finitely generated subgroup of G is a finite extension of a nilpotent group of class at most $m + 1$, and the conclusion of the theorem follows from Lemma 2.